Answer on Question #51712 - Math - Vector Calculus

$\nabla(x/r^3) = ?$, $\quad \nabla(\vec{r}/r^3) = ?$

where

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

<u>**Solution**</u>:

$\nabla(x/r^3)$:

$$\nabla\left(\frac{x}{r^3}\right) = \left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right]\vec{\imath} + \left[\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right]\vec{\jmath} + \left[\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)\right]\vec{k}$$

Since $r = \sqrt{x^2 + y^2 + z^2}$, we obtain

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{\frac{\partial}{\partial x}(x)}{r^3} + x\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) =$$

$$= \frac{1}{r^3} + x\frac{\partial}{\partial x}\left(\frac{1}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3}\right) =$$

$$= \frac{1}{r^3} - x\frac{3}{2}\frac{\frac{\partial}{\partial x}(x^2 + y^2 + z^2)}{\left(\sqrt{x^2 + y^2 + z^2}\right)^5} = \frac{1}{r^3} - \frac{3x^2}{\left(\sqrt{x^2 + y^2 + z^2}\right)^5} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

Similarly

$$\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = x\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) = -\frac{3yx}{r^5}$$

$$\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) = x\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right) = -\frac{3zx}{r^5}$$

Substituting these into the first equation we obtain

$$\nabla\left(\frac{x}{r^3}\right) = \left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right]\vec{\imath} + \left[\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right]\vec{\jmath} + \left[\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)\right]\vec{k} = \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)\vec{\imath} - \frac{3yx}{r^5}\vec{\jmath} - \frac{3zx}{r^5}\vec{k} =$$

$$= \frac{\vec{\imath}}{r^3} - 3x\frac{\vec{r}}{r^5}$$

$\nabla(\vec{r}/r^3)$:

$$\nabla\left(\frac{\vec{r}}{r^3}\right) = \left[\nabla\left(\frac{x}{r^3}\right)\right]\vec{\imath} + \left[\nabla\left(\frac{y}{r^3}\right)\right]\vec{\jmath} + \left[\nabla\left(\frac{z}{r^3}\right)\right]\vec{k}$$

Since

$$\nabla\left(\frac{x}{r^3}\right) = \frac{\vec{\imath}}{r^3} - 3x\frac{\vec{r}}{r^5}$$

$$\nabla\left(\frac{y}{r^3}\right) = \frac{\vec{\jmath}}{r^3} - 3y\frac{\vec{r}}{r^5}$$

$$\nabla\left(\frac{z}{r^3}\right) = \frac{\vec{k}}{r^3} - 3z\frac{\vec{r}}{r^5}$$

Therefore

$$\nabla\left(\frac{\vec{r}}{r^3}\right) = \left[\nabla\left(\frac{x}{r^3}\right)\right]\vec{\imath} + \left[\nabla\left(\frac{y}{r^3}\right)\right]\vec{\jmath} + \left[\nabla\left(\frac{z}{r^3}\right)\right]\vec{k} =$$

$$= \left[\frac{\vec{\imath}}{r^3} - 3x\frac{\vec{r}}{r^5}\right]\vec{\imath} + \left[\frac{\vec{\jmath}}{r^3} - 3y\frac{\vec{r}}{r^5}\right]\vec{\jmath} + \left[\frac{\vec{k}}{r^3} - 3z\frac{\vec{r}}{r^5}\right]\vec{k} =$$

$$= \frac{1}{r^3} - 3x\frac{x}{r^5} + \frac{1}{r^3} - 3y\frac{y}{r^5} + \frac{1}{r^3} - 3z\frac{z}{r^5} = \frac{3}{r^3} - 3\frac{x^2+y^2+z^2}{r^5} =$$

$$= \frac{3}{r^3} - 3\frac{r^2}{r^5} = \frac{3}{r^3} - \frac{3}{r^3} = 0$$

<u>Answer:</u> $\nabla\left(\frac{x}{r^3}\right) = \frac{\vec{\imath}}{r^3} - 3x\frac{\vec{r}}{r^5}$, $\nabla\left(\frac{\vec{r}}{r^3}\right) = 0$.