Answer on Question #51481 – Math – Differential Equations

Question

Given the following monotonically transformed utility function faced by the consumer:

$$\ln U(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$$

The price of good x_1 is p_1 and the price of good x_2 is p_2.

1) Construct the corresponding Lagrangian function

2) Derive the optimal demand (Marshallian demand) function for x_1 and for x_2.

Solution

1) Lagrangian function:

Let I be income, so we can write the budget constraint:

$$I = x_1 p_1 + x_2 p_2$$

$$I - x_1 p_1 - x_2 p_2 = 0$$

We need to solve the problem of optimization:

$$\max(\ln U(x_1, x_2))$$

So we can construct the Lagrangian function:

$$L(x_1, x_2, \lambda) = \alpha \ln x_1 + \beta \ln x_2 + \lambda(I - x_1 p_1 - x_2 p_2)$$

2) Marshallian demand:

$$[x_1]: \frac{\partial L}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0 \quad \Rightarrow \quad \frac{\alpha}{x_1} = \lambda p_1 \quad \Rightarrow \quad x_1 p_1 = \frac{\alpha}{\lambda}$$

$$[x_2]: \frac{\partial L}{\partial x_2} = \frac{\beta}{x_2} - \lambda p_2 = 0 \quad \Rightarrow \quad \frac{\beta}{x_2} = \lambda p_2 \quad \Rightarrow \quad x_2 p_2 = \frac{\beta}{\lambda}$$

$$[\lambda]: \frac{\partial L}{\partial \lambda} = I - x_1 p_1 - x_2 p_2 = 0$$

After substitution in the budget constraint:

$$I - \frac{\alpha}{\lambda} - \frac{\beta}{\lambda} = 0 \quad \Rightarrow \quad \frac{\alpha + \beta}{\lambda} = I$$

$$\frac{1}{\lambda} = \frac{I}{\alpha + \beta}$$

Substituting back in the original first-order conditions give us Marshallian demand function for x_1 and for x_2:

$$x^* = \left(x_1 = \frac{\alpha}{\alpha + \beta} \frac{I}{p_1};\ x_2 = \frac{\beta}{\alpha + \beta} \frac{I}{p_2}\right)$$