Answer on Question #51339 – Math – Differential Calculus | Equations

Separate the following partial differential equation into a set of three ODEs by the method of separation of variables :

d^2 u/dt^2 = c^2 [d^2 u/dr^2 + 1/r * du/dr + 1/r^2 * d^2/d(thita)^2]

Solution

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$

Assume that $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

So $R(r)\Theta(\theta)T''(t) = c^2[\left(R'' + \frac{1}{r}R'\right)\Theta(\theta)T(t) + \frac{1}{r^2}\Theta''(\theta)R(r)T(t)] \rightarrow$ **divide both sides by** $c^2R(r)\Theta(\theta)T(t) \rightarrow$

$$\rightarrow \frac{1}{c^2}\frac{T''}{T} = \left(R'' + \frac{1}{r}R'\right)\frac{1}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}$$

The right side of this equation does not depend on T**, hence the left side of this equation must be constant.**

Thus, $\frac{1}{c^2}\frac{T''}{T} = \left(R'' + \frac{1}{r}R'\right)\frac{1}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda.$

$$\left(R'' + \frac{1}{r}R'\right)\frac{1}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda \rightarrow \quad -\frac{\Theta''}{\Theta} = \left(R'' + \frac{1}{r}R'\right)\frac{r^2}{R} - \lambda r^2$$

Because each side only depends on one independent variable, both sides of this equation must be constant. This gives the second separation constant, which we call μ.

The equation with respect to Θ **can then be written as**

$$\Theta'' + \mu\Theta = 0$$

And equation with respect to R:

$$\left(R'' + \frac{1}{r}R'\right)\frac{r^2}{R} - \lambda r^2 = \mu \to \quad r^2R'' + rR' - (\lambda r^2 + \mu)R = 0$$

Finally we have 3 ODEs:

$$T'' - \lambda c^2 T = 0;$$

$$\Theta'' + \mu\Theta = 0;$$

$$r^2R'' + rR' - (\lambda r^2 + \mu)R = 0.$$