Answer on Question #51248 - Math - Differential Equations

Solve: $z(p-q) = z^2 + (x+y^2)$

Solution:

p denotes $\frac{\partial z}{\partial x}$, and q denotes $\frac{\partial z}{\partial y}$. Therefore,

$$z\frac{\partial z}{\partial x} - z\frac{\partial z}{\partial y} = z^2 + (x+y^2)$$

The general solution of this differential equation is given by

$$F[u_1(x,y,z), u_2(x,y,z)] = 0,$$

where $F(u_1, u_2)$ is continuously differentiable function, $u_1(x,y,z)$ and $u_2(x,y,z)$ are two independent first integrals of autonomous system

$$\frac{dx}{z} = -\frac{dy}{z} = \frac{dz}{z^2 + (x+y^2)}$$

Let's first consider the following equation

$$\frac{dx}{z} = -\frac{dy}{z}$$

$$d(x+y) = 0$$

This gives one of the first integrals

$$u_1(x,y,z) = x + y$$

Let's now consider the another equation

$$-\frac{dy}{z} = \frac{dz}{z^2 + (x+y^2)}$$

$$z^2 + (x+y^2) = -zz'$$

Using $u_1 = x + y$, we obtain

$$z^2 + u_1 - y + y^2 = -\frac{1}{2}(z^2)'$$

Let's first find the solution of homogenous equation:

$$z^2 = -\frac{1}{2}(z^2)'$$

Its solution is given by

$$z_p^2 = u_2 e^{-2y},$$

where u_2 is some constant.

The particular solution is given by

$$z_p^2 = -1 - u_1 + 2y - y^2$$

Indeed

$$z_p^2 + \frac{1}{2}\left(z_p^2\right)' = -1 - u_1 + 2y - y^2 + \frac{1}{2}(2 - 2y) = -u_1 + y - y^2$$

Therefore, the general solution is given by

$$z^2 = u_2 e^{-2y} - 1 - u_1 + 2y - y^2$$

which is equivalent to

$$u_2 = e^{2y}(z^2 + 1 + u_1 - 2y + y^2) = e^{2y}(z^2 + 1 + (x + y) - 2y + y^2) =$$

$$= e^{2y}(z^2 + 1 + x - y + y^2)$$

This gives another first integral. Therefore, the solution of the initial equation is given by

$$F[x + y, e^{2y}(z^2 + 1 + x - y + y^2)] = 0$$

<u>Answer:</u> $F[x + y, e^{2y}(z^2 + 1 + x - y + y^2)] = 0$, where $F(u_1, u_2)$ is continuously differentiable function.