Answer on Question #51034 – Math – Differential Calculus

Question:

a) Find $f(y)$ so that equation $f(y)dx - zxdy - xy\ln y\,dz = 0$ is integrable. Also obtain the corresponding integral using Natani's method.

b) Find the differential equation of the family of surfaces $\Phi(z(x+y)^2, x^2 - y^2) = 0$

Solution:

a) Equation $f(y)dx - zxdy - xy\ln y\,dz = 0$ is integrable, if

$$\boldsymbol{X} \cdot rot\,\boldsymbol{X} = 0$$

$$\boldsymbol{X} = (f(y), -zx, -xy\ln y)$$

Let's calculate $rot\boldsymbol{X}$

$$rot\boldsymbol{X} = \boldsymbol{\nabla} \times \boldsymbol{X} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(y) & -zx & -xy\ln y \end{vmatrix}$$

$$= \boldsymbol{i}\left(\frac{\partial}{\partial y}(-xy\ln y) - \frac{\partial}{\partial z}(-xz)\right) - \boldsymbol{j}\left(\frac{\partial}{\partial x}(-xy\ln y) - \frac{\partial}{\partial z}f(y)\right)$$

$$+ \boldsymbol{k}\left(\frac{\partial}{\partial x}(-zx) - \frac{\partial}{\partial y}f(y)\right) = (-x\ln y)\boldsymbol{i} + (y\ln y)\boldsymbol{j} + \big(-z - f'(y)\big)\boldsymbol{k}$$

Then

$$\boldsymbol{X} \cdot rot\,\boldsymbol{X} = -x\ln y \cdot f(y) + xy\ln y\, f'(y) = 0$$

Thus $x\ln y \cdot (yf'(y) - f(y)) = 0$, hence $yf'(y) - f(y) = 0$

The solution of the last equation is determined by

$$\frac{df}{f} = \frac{dy}{y} \quad \rightarrow \ln|f(y)| = \ln|y| + \ln|A| \quad \rightarrow \quad f(y) = A \cdot y$$

where A is a real constant. Finally, equation takes the form

$$Aydx - zxdy - xy\ln y\,dz = 0$$

Using Natani's method we firstly put $z = const$ and solve the obtained equation

$$Aydx = zxdy \quad \rightarrow A\frac{dx}{x} = z\frac{dy}{y} \quad \rightarrow \quad A\ln x = z\ln y + F(z)$$

Now we let $x = 1$. Then equation takes the form

$$-zdy - y\ln y\,dz = 0$$

and

$F(z) = -\mathrm{z} \cdot \ln y$. Thus,

$$zdy + y \ln y \, dz = 0 \quad \rightarrow \frac{dy}{y\ln y} + \frac{dz}{z} = 0 \ \rightarrow \ln|\ln y| = -\ln|z| + \ln|C| \rightarrow$$

$$\ln y = \frac{C}{z}$$

Therefore,

$F(z) = -z\ln y = -z \cdot \frac{C}{z} = -C = K$ (another constant).

Finally,

$$A \ln x = z \ln y + K$$

where K is an arbitrary real constant.

b) The general solution of first-order differential equation

$$\Phi(z(x+y)^2, x^2 - y^2) = 0$$

implies that

$$z(x+y)^2 = c_1 = const$$

$$x^2 - y^2 = c_2 = const$$

After differentiating these both equation we get

$$dz = -2z \cdot \frac{dx + dy}{x + y}$$

$$xdx = ydy$$

First-order differential equation

$$a \cdot \frac{\partial z}{\partial x} + b \cdot \frac{\partial z}{\partial y} = c$$

is equivalent to

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}$$

From $xdx = ydy$ and previous equation, we get

$$\frac{dx}{y} = \frac{dy}{x} \quad \rightarrow \quad a = y, \ \ b = x$$

Putting $dx = \frac{y}{x} dy$ into $dz = -2z \cdot \frac{dx+dy}{x+y}$, we get

$$dz = -2z \cdot \frac{dy}{x}$$

Thus

$$\frac{dz}{c} = -\frac{2z}{c} \cdot \frac{dy}{x} = \frac{dy}{b} = \frac{dy}{x} \quad \to \quad c = -2z$$

Finally,

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{-2z}$$

or

$$y \cdot \frac{\partial z}{\partial x} + x \cdot \frac{\partial z}{\partial y} = -2z$$

Answer: $y \cdot \frac{\partial z}{\partial x} + x \cdot \frac{\partial z}{\partial y} = -2z$