Answer on Question #50907 – Math – Differential Calculus | Equations

a) Given:

differential equation $x^3p^2 + x^2yp + a^3 = 0$

Task: Solve it and also obtain its singular solution, if it exists

Solution:

we understand that $p = \frac{dy}{dx}$ $p^2 = \frac{d^2y}{dx^2}$

so $x^3y'' + x^2yy' + a^3 = 0$

$y'' + \frac{y}{x}y' + \frac{a^3}{x^3} = 0$

$y'' + \frac{1}{x}\left(\frac{y^2}{2}\right)' + \frac{a^3}{x^3} = 0$ is a nonlinear differential equation

It can be solved using power series.

Other case is $x^3\left(\frac{dy}{dx}\right)^2 + x^2y\frac{dy}{dx} + a^3 = 0$

$D = x^4y^2 - 4x^3a^3$

the solution is detrermined by $\frac{dy}{dx} = \frac{-x^2y \pm \sqrt{x^4y^2 - 4x^3a^3}}{x} = -xy \pm \sqrt{x^2y^2 - 4xa^3}$.

Let $xy = c$ where $c = const$

then $y' = -\frac{c}{x^2}$ $y'' = \frac{2c}{x^3}$

we obtain

$-\frac{c}{x^2} = -c \pm \sqrt{c^2 - 4xa^3}$

this solution is singular $\Leftrightarrow$ $x = 0$

b) Given:

$EI\left(\frac{d^2y}{dx^2}\right) = Py - \frac{1}{2}Wx^2$

W is beam uniform load

P is force

E is the modulus of elasticity

I is the moment of inertia

Show:

that the elastic curve for the beam with conditions y = 0 and dy/dx = 0 at x = 0 , is given by $y = \frac{W(1 - ch(nx))}{Pn^2} + \frac{Wx^2}{2P}$

where $n^2 = \frac{P}{EI}$

Solution

$$y'' = -\frac{W}{P}ch(nx) + \frac{W}{P}$$

then our equation will be

$$EI(-\frac{W}{P}ch(nx) + \frac{W}{P}) = \frac{W}{n^2} - \frac{W}{n^2}ch(nx) + \frac{1}{2}Wx^2 - \frac{1}{2}Wx^2$$

$$-\frac{WEI}{P}ch(nx) + \frac{WEI}{P} = \frac{WEI}{P} - \frac{WEI}{P}ch(nx)$$

converse to identity

so $y = \frac{W(1 - ch(nx))}{Pn^2} + \frac{Wx^2}{2P}$ is the elastic curve for the beam

c) Given:

$$\frac{du}{dt} - 4\frac{d^2u}{dx^2} = 0 \qquad (1)$$

$$\begin{cases} u\big|_{x=0} = 0 \\ u\big|_{x=5} = 0 \end{cases} \qquad (2)$$

$$u\big|_{t=0} = x \qquad (3)$$

Solution:

(2) are homogeneous so we use Fourier Method

Step 1

the partial solutions of (1),(2):

$$u(x,t) = X(x) \cdot T(t)$$

$$(1) \Rightarrow XT' - 4X''T = 0$$

$$\frac{T'}{4T} = \frac{X''}{X} = -\lambda = const$$

$$X'' + \lambda X = 0$$

1) $\lambda = 0 \quad \Leftrightarrow \quad X = ax + b$

$$(2) \Rightarrow \begin{matrix} X(0) = 0 \\ X(5) = 0 \end{matrix} \Rightarrow \begin{matrix} b = 0 \\ 5a + b = 0 \end{matrix} \quad \begin{matrix} a = 0 \\ b = 0 \end{matrix} \Rightarrow X \equiv 0$$

2) $\lambda > 0$

the characteristic equation

$$p^2 + \lambda = 0$$

$$X(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

$$\begin{matrix} X(0) = 0 \\ X(5) = 0 \end{matrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 \sin(\sqrt{\lambda} x) = 0 \end{matrix} \Rightarrow \begin{matrix} c_2 \neq 0 \\ \sin(\sqrt{\lambda} x) = 0 \end{matrix}$$

$$\sqrt{\lambda} = \frac{\pi k}{5} \qquad k \in N$$

$X(x) = X_k(x) = \sin(\frac{\pi k}{5} x)$ are the system of eigen functions

$\{X_k(x), k \geq 1\}$ is a complete and ortogonal system on $0 < x < 5$

$$\|X_k\|_2^2 = \int_0^5 \sin^2(\frac{\pi k}{5} x) dx = \frac{5}{2} - \frac{5}{4\pi k} \sin(2\pi k) = \frac{5}{2}$$

$$T' + 4\lambda_k T = 0$$

$$p + 4\lambda_k = 0$$

$$p = -4\lambda_k$$

$T = T_k(t) = c_k e^{-4\lambda_k t}$ where $\lambda_k = \frac{\pi^2 k^2}{25}$

so the the partial solutions of (1),(2) are:

$$u_k(x,t) = c_k e^{-4\lambda_k t} \cdot \sin(\frac{\pi k}{5} x)$$

Step 2

Solution of (1),(2),(3):

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} c_k e^{-4\frac{\pi^2 k^2}{25} t} \cdot \sin(\frac{\pi k}{5} x) \qquad 0 < x < 5, \ t > 0$$

$$(3) \Rightarrow \quad u\big|_{t=0} = \sum_{k=1}^{\infty} c_k \sin(\frac{\pi k}{5} x) = x$$

$$c_k = \frac{(x, \sin(\frac{\pi k}{5} x))}{\|X_k\|_2^2} = \frac{2}{5} \int_0^5 x \cdot \sin(\frac{\pi k}{5} x) dx = \frac{2}{5} [(-\frac{5x}{\pi k} \cos(\frac{\pi k}{5} x) + \frac{25}{\pi^2 k^2} \sin(\frac{\pi k}{5} x))\Big|_0^5] =$$

$$= \frac{2}{5} [-\frac{25}{\pi k} \cos(\pi k) + \frac{25}{\pi^2 k^2} \sin(\pi k)] = -\frac{10}{\pi k} (-1)^k = \frac{10}{\pi k} (-1)^{k+1}$$

Answer:

$$u(x,t) = \sum_{k=1}^{\infty} \frac{10}{\pi k} (-1)^{k+1} \cdot e^{-4\frac{\pi^2 k^2}{25} t} \cdot \sin(\frac{\pi k}{5} x)$$