Answer on Question #50904 – Math – Differential Calculus | Equations

a) Find the surface which intersects the surfaces of the system $z\ (x+y)\ =\ c\ (3z+1)$ orthogonally and which passes through the circle $x^2+\ y^2\ =\ 1, z\ =\ 1$.

b) Show that the complete integral of $z\ =\ px\ +\ qy\ -\ 2p\ -\ 3q$ represents all possible planes through the points (2, 3, 0).

c) Find the values of n for which the equation $(n-1)^2\ u_{xx}-\ y^{2n}\ u_{yy}\ =\ ny^{2n-1}u_y$ is

i) parabolic **ii)** hyperbolic.

Solution

a) The given system of surfaces is given by

$$f(x,y,z)=\frac{z(x+y)}{3z+1}=C. \qquad (1)$$

$$\frac{\partial f}{\partial x}=\frac{z}{3z+1},\frac{\partial f}{\partial y}=\frac{z}{3z+1},\frac{\partial f}{\partial z}=(x+y)\frac{(3z+1)-3z}{(3z+1)^2}=\frac{(x+y)}{(3z+1)^2}.$$

The required orthogonal surface is solution of

$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}$ or $p\frac{z}{3z+1}+q\frac{z}{3z+1}=\frac{(x+y)}{(3z+1)^2}$ or $z(3z+1)p+z(3z+1)q=x+y.$ **(2)**

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{z(3z+1)}=\frac{dy}{z(3z+1)}=\frac{dz}{x+y}. \qquad (3)$$

Taking the first two fractions of (3), we get

$$dx-dy=0\ so\ that\ x-y=C_1. \qquad (4)$$

Choosing $x, y, -z(3z+1)$ as multipliers, each fraction of (3)

$$=\frac{xdx+ydy-z(3z+1)dz}{0}\rightarrow xdx+ydy-2z^2dz-zdz.$$

Integrating, $\frac{1}{2}x^2+\frac{1}{2}y^2-3\left(\frac{z^3}{3}\right)-\frac{1}{2}z^2=\frac{1}{2}C_2$ or $x^2+y^2-2z^3-z^2=C_2$. **(5)**

Hence any surface which is orthogonal to (1) has equation of the form

$x^2+y^2-2z^3-z^2=\phi(x-y),$ **(6)**

where ϕ being arbitrary function .

In order to get the desired surface passing through the circle $x^2+\ y^2\ =\ 1, z\ =\ 1$ we must choose $\phi(x-y)=-2$. Thus, the required particular surface is

$$x^2+y^2-2z^3-z^2=-2.$$

b) Given that $z\ =\ px\ +\ qy\ -\ 2p\ -\ 3q,$ **(7)**

which is of the form $z\ =\ px\ +\ qy+f(p,q)$ and so its complete integral is

$z = ax + by - 2a - 3b, a, b$ being arbitrary constants. **(8)**

Since (8) is a linear equation in x, y, z, it follows that (8) represents planes for various values of a and b. Again putting $x = 2, y = 3, z = 0$ in (8) we have

$$0 = 2a + 3b - 2a - 2b, i.e. 0 = 0,$$

showing that the coordinates of the point (2, 3, 0) satisfy (8). Hence the complete integral (8) of (7) represents all possible planes passing through the point (2, 3, 0).

c) i) The equation is parabolic, if

$$D = B^2 - 4AC = 0.$$

$$0 - 4(n-1)^2(-y^{2n}) = 0.$$

So $n = 1$. We have:

$$-y^2 u_{yy} = yu_y.$$

Answer: 1.

ii) The equation is hyperbolic, if

$$D = B^2 - 4AC > 0.$$

$$0 - 4(n-1)^2(-y^{2n}) = \left(2y^n(n-1)\right)^2 > 0.$$

So it is true for $n \neq 1$.

Answer: $(-\infty; \mathbf{1}) \cup (\mathbf{1}; \infty)$**.**