Answer on Question #50900 – Math – Differential Calculus | Equations

a) Given:

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2} = \frac{dz}{z(x-y)}$$

Find: the integral curves of the equations

Solution

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2} \quad \Rightarrow \quad dx = dy - \frac{yz}{x^2 - y^2}dy \quad (1)$$

$$\frac{dy}{x^2 - y^2} = \frac{dz}{z(x-y)} \quad \Rightarrow \quad dy = \frac{x+y}{z}dz \quad (2)$$

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dz}{z(x-y)} \quad \Rightarrow \quad dx = \frac{x+y}{z}dz - \frac{y}{x-y}dz \quad (3),$$

(2) and (3) $\Rightarrow$

$$dx = dy - \frac{y}{x-y}dz \quad (4) \Rightarrow$$

(1) and (4) $\Rightarrow$

$$\frac{y}{x-y}dz = \frac{yz}{x^2 - y^2}dy$$, divide both sides by $\frac{y}{x-y}$

$\Rightarrow \quad dz = \frac{z}{x+y}dy$, separate variables

$$\frac{dz}{z} = \frac{dy}{x+y}$$

Integrate both sides

$$Ln|z| = Ln|x+y| + Ln|c|$$

Apply properties of logarithmic function

$$|z| = |c(x+y)|,$$

leave signs of absolute values by means of choice of c,

$z = c(x+y)$, where c is arbitrary real constant.

Answer: $z = c(x+y)$, c=const

b) Given:

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)}$$

Find: the integral curves of the equations

Solution

$$\frac{dx}{x^2+y^2}=\frac{dy}{2xy} \qquad \Rightarrow \qquad \frac{dy}{dx}=\frac{2xy}{x^2+y^2}$$

it is a homogeneous differential equation
so we replace $y=ux$

$$\frac{dy}{dx}=(ux)'=u+u'x=\frac{2ux^2}{x^2+u^2x^2}=\frac{2u}{1+u^2}$$

$$u'x=\frac{2u}{1+u^2}-u=\frac{2u-u-u^3}{1+u^2}=\frac{u-u^3}{1+u^2}$$

$$\frac{du}{dx}x=\frac{u-u^3}{1+u^2} \qquad \Rightarrow \qquad \frac{1+u^2}{u-u^3}du=\frac{dx}{x}$$

and now we decompose on simple fractions

$$\frac{1+u^2}{u(1-u)(1+u)}=\frac{A}{u}+\frac{B}{1-u}+\frac{C}{1+u}$$

$$\frac{A}{u}+\frac{B}{1-u}+\frac{C}{1+u}=\frac{A-Au^2+Bu+Bu^2+Cu-Cu^2}{u(1-u)(1+u)}=\frac{1+u^2}{u(1-u)(1+u)} \qquad \Rightarrow$$

$$\begin{cases}-A+B-C=1\\ B+C=0\\ A=1\end{cases} \Rightarrow \begin{cases}B-C=2\\ B=-C\\ A=1\end{cases} \Rightarrow \begin{cases}2B=2\\ B=-C\\ A=1\end{cases} \Rightarrow \begin{cases}A=1\\ B=1\\ C=-1\end{cases}$$

$$\Rightarrow \quad \int(\frac{1}{u}+\frac{1}{1-u}-\frac{1}{1+u})du=\int\frac{dx}{x}$$

$$Ln|u|-Ln|1-u|-Ln|1+u|=Ln|x|+Ln|c|$$

$$\frac{u}{1-u^2}=xc$$

and now we return to y

$$\frac{y/x}{1-y^2/x^2}=xc \Rightarrow$$

$$\frac{y}{x^2-y^2}=c \Rightarrow \qquad x=\sqrt{\frac{y}{c}+y^2}$$

$$\frac{dy}{2xy}=\frac{dz}{z(x+y)} \qquad \Rightarrow \qquad \frac{\sqrt{\frac{y}{c}+y^2}+y}{2y\sqrt{\frac{y}{c}+y^2}}dy=\frac{dz}{z}$$

$$\int (\frac{dy}{2y} + \frac{dy}{2\sqrt{\frac{y}{c} + y^2}}) = \int \frac{dz}{z} \qquad (1)$$

At first we calculate the integral

$$\int \frac{dy}{\sqrt{\frac{y}{c} + y^2}} = \int \frac{dy}{\sqrt{(y + \frac{1}{2c})^2 - \frac{1}{4c^2}}} = Ln\left| y + \frac{1}{2c} + \sqrt{\frac{y}{c} + y^2} \right|$$

So we obtain

$$(1) \quad \Rightarrow \quad Ln|y| + Ln\left| y + \frac{1}{2c} + \sqrt{\frac{y}{c} + y^2} \right| = Ln|z|^2$$

$$z^2 = y(y + \frac{1}{2c} + \sqrt{\frac{y}{c} + y^2})$$

Answer: $z^2 = y(x + y + \frac{1}{2c})$, $\qquad c = const$

c) Given:

partial differential equation $(x - y)y^2 p + (y - x)x^2 q = (x^2 + y^2)z$

where $\qquad p = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y}$

Find:

the integral surfaces through the curve $xz = a^2$, $\quad y = 0$

Solution

$$(x - y)y^2 \frac{\partial z}{\partial x} + (y - x)x^2 \frac{\partial z}{\partial y} = (x^2 + y^2)z, \quad z(x,0) = \frac{a^2}{x}$$

$$\frac{dx}{(x - y)y^2} = \frac{dy}{(y - x)x^2} \text{ or } \frac{dz}{(x^2 + y^2)z} = \frac{dy}{(y - x)x^2} \text{ or } \frac{dz}{(x^2 + y^2)z} = \frac{dx}{(x - y)y^2}$$

In equation $\frac{dx}{(x - y)y^2} = \frac{dy}{(y - x)x^2}$ multiply both sides by $(x - y)$:

$$\frac{dx}{y^2} = -\frac{dy}{x^2}$$

So $x^2 dx + y^2 dy = 0$, which gives $x^3 + y^3 = C_1$.

We put $y = 0$ and $z = \frac{a^2}{x}$ to the partial differential equation, which yields

$$-x^3 \frac{\partial z}{\partial y} = x^2 z$$

$$-x^3 \frac{\partial z}{\partial y} = xa^2$$

$$x^2 \frac{\partial z}{\partial y} = -a^2$$

$$\frac{\partial z}{\partial y} = -\frac{a^2}{x^2}$$

$$z = -\frac{a^2}{x^2} y + \varphi(x)$$

$$xz = -\frac{a^2}{x} y + x\varphi(x)$$

For $y = 0$ put $-\frac{a^2}{x} \cdot 0 + x\varphi(x) = a^2$, hence $\varphi(x) = \frac{a^2}{x}$

$$z = -\frac{a^2}{x^2} y + \frac{a^2}{x}$$

Answer: $z = -\frac{a^2}{x^2} y + \frac{a^2}{x}$