Answer on Question #50899-Math-Differential Calculus-Equations

Apply the method of variations of parameters to solve the following differential equations:

a) $x^2\, y'' + x\, y' - y = x^2\, e^x$

b) $y'' + a^2\, y = cosec\ ax$

c) Solve the equation $\frac{d^2y}{dx^2} - cot\, x \frac{dy}{dx} - sin^2 xy = cos\, x - cos^3\, x$ by changing the independent variable.

Solution

a) $x^2\, y'' + x\, y' - y = 0$

$$y = x^n \to n(n-1) + n - 1 = 0 \to n = \pm 1.$$

$$y = c_1 x + c_2 \frac{1}{x}.$$

The Wronskian is given by

$$W = \begin{bmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{bmatrix} = -\left(\frac{1}{x} + \frac{1}{x}\right) = -\frac{2}{x} \neq 0.$$

We seek the solution in the form $y = y_1 v_1 + y_2 v_2$

$$v_1 = \int \left(-\frac{y_2 R}{W}\right) dx = -\int \frac{\frac{1}{x} \cdot x^2\, e^x}{-\frac{2}{x}} dx = \frac{e^x}{2}(x^2 - 2x + 2) + c_1.$$

$$v_2 = \int \left(\frac{y_1 R}{W}\right) dx = \int \frac{x \cdot x^2\, e^x}{-\frac{2}{x}} dx = -\frac{e^x}{2}(x^4 - 4x^3 + 12x^2 - 24x + 24) + c_2.$$

So,

$$y = x\left[\frac{e^x}{2}(x^2 - 2x + 2) + c_1\right] + \frac{1}{x}\left[-\frac{e^x}{2}(x^4 - 4x^3 + 12x^2 - 24x + 24) + c_2\right].$$

$$y = c_1 x + c_2 \frac{1}{x} + e^x\left[x^2 - 5x + 12 - \frac{12}{x}\right]$$

b) $y'' + a^2\, y = 0.$

$$y = c_2 \cos ax + c_1 \sin ax.$$

The Wronskian is given by

$$W = \begin{bmatrix} \sin ax & \cos ax \\ a\cos ax & -a\sin ax \end{bmatrix} = -a(\sin^2 ax + \cos^2 ax) = -a \neq 0.$$

We seek the solution in the form $y = y_1 v_1 + y_2 v_2$

$$v_1 = \int \left(-\frac{y_2 R}{W}\right) dx = -\int \frac{\cos ax \cdot\ cosec\ ax}{-a} dx = \frac{1}{a^2} \ln|\sin ax| + c_1.$$

$$v_2 = \int \left(\frac{y_1 R}{W}\right) dx = \int \frac{\sin ax \cdot\ cosec\ ax}{-a} dx = \frac{x}{a} + c_2.$$

So,

$$y = \cos ax \left[\frac{x}{a} + c_2\right] + \sin ax \left[\frac{1}{a^2} \ln|\sin ax| + c_1\right].$$

c) Comparing the given equation with $y'' + Py' + Qy = R$, we have $P = -cot\ x$, $Q = -sin^2 x$ and $R = cos\ x\ -\ cos^3\ x = \cos x \sin^2 x$. (1)

Choose z such that $\left(\frac{dz}{dx}\right)^2 = sin^2 x$ and $\frac{dz}{dx} = \sin x$. (2)

Integrating, $z = \int \sin x\, dx = -\cos x$. (3)

Now changing the independent variable from x to z by using relation (3), the given equation is transformed into

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1, \quad (4)$$

where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x + (-cot\ x)(\sin x)}{sin^2 x} = 0$, by (1) and (2)

$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-sin^2 x}{sin^2 x} = -1$, $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x \sin^2 x}{sin^2 x} = \cos x = -z.$ (5)

From (5), $\frac{d^2y}{dz^2} - y = -z$, or $(D_1^2 - 1)y = -z$, where $D_1 = \frac{d}{dz}$.

Its auxiliary equation is $m^2 - 1 = 0$ so that $m = \pm 1$. General solution of (5) is

$c_1 e^z + c_2 e^{-z} = c_1 e^{-\cos x} + c_2 e^{\cos x}$, by (3)

Partial solution is

$$\frac{1}{D_1^2 - 1}(-z) = \frac{1}{1 - D_1^2} z = (1 - D_1^2)^{-1} z = (1 + D_1^2 + \cdots) z = z = -\cos x.$$

Hence the required solution is

$$y = c_1 e^{-\cos x} + c_2 e^{\cos x} - \cos x.$$