Answer on Question #50894 – Math – Differential Calculus | Equations

a) Write a suitable form of particular solution for solving the equation

$$y'' + 3y' + 2y = e^{x}(x^2 + 1)\, sin\, 2x + 3e^{-x} \,cos\, x + 4e^{x}$$

by the method of undetermined coefficients.

b) Verify that e^x and xe^x are solutions of the homogeneous equation corresponding to the equation

$$y'' - 2y' + y = \frac{e^x}{1 + x^2}, -\infty < x < \infty$$

and find the general solution using the method of variation of parameters.

c) If $y_1 = 2x + 2$ and $y_2 = -\frac{x^2}{2}$ are the solutions of

$$xy' + y' - \frac{y'^2}{2}$$

then are the constant multiples $c_1 y_1$ and $c_2 y_2$, where c_1 and c_2 are arbitrary, also solutions of the given differential equation? Is the sum $y_1 + y_2$ a solution?

Solution

a) We first consider the solutions to the corresponding homogeneous equation. Using the characteristic equation $m^2 + 3m + 2 = 0 \to m_1 = -2, m_2 = -1$, we have solutions $y_1 = e^{-2x}$ and $y_2 = e^{-x}$. Thus we have the form

$$Y = (Ax^2 + Bx + C)e^x \sin 2x + (Dx^2 + Ex + F)e^x \cos 2x + Ge^{-x} \cos x + He^{-x} \sin x + Ie^x.$$

b) The homogeneous equation corresponding to the equation

$$y'' - 2y' + y = \frac{e^x}{1 + x^2}, -\infty < x < \infty$$

Is

$$y'' - 2y' + y = 0, -\infty < x < \infty$$

$$\frac{d^2}{dx^2}(e^x) = e^x, \frac{d}{dx}(e^x) = e^x.$$

So,

$$y'' - 2y' + y = e^x - 2e^x + e^x \equiv 0.$$

And we see that e^x is a solution.

$$\frac{d^2}{dx^2}(xe^x) = (x + 2)e^x, \frac{d}{dx}(xe^x) = (x + 1)e^x.$$

So,

$$y'' - 2y' + y = (x + 2)e^x - 2(x + 1)e^x + xe^x \equiv 0.$$

And we see that xe^x is a solution.

$$y = c_1 e^x + c_2 x e^x.$$

The Wronskian is given by

$$W = \begin{bmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{bmatrix} = (x+1)e^{2x} - xe^{2x} = e^{2x} \neq 0.$$

We seek the solution in the form $y = y_1 v_1 + y_2 v_2$

$$v_1 = \int \left(-\frac{y_2 R}{W}\right) dx = -\int \frac{xe^x \cdot \frac{e^x}{1 + x^2}}{e^{2x}} dx = \ln \frac{1}{\sqrt{1 + x^2}} + c_1.$$

$$v_2 = \int \left(\frac{y_1 R}{W}\right) dx = \int \frac{e^x \cdot \frac{e^x}{1 + x^2}}{e^{2x}} dx = \tan^{-1} x + c_2.$$

So,

$$y = e^x \left[\ln \frac{1}{\sqrt{1 + x^2}} + c_1\right] + xe^x [\tan^{-1} x + c_2].$$

c) The constant multiples $c_1 y_1$ and $c_2 y_2$ or the sum $y_1 + y_2$ are solutions **if and only if** an equation is a linear. But

$$xy' + y' - \frac{y'^2}{2}$$

contains nonlinear element (quadratic $\frac{y'^2}{2}$). Thus $c_1 y_1$ and $c_2 y_2$ or the sum $y_1 + y_2$ are not solutions.