Answer on Question #50830 – Math – Integral Calculus

Question. Find indefinite integral

$$\int \sec u \, du$$

Solution. Recall that by definition $\sec u = \frac{1}{\cos u}$. Thus

$$\int \sec u \, du = \int \frac{du}{\cos u}.$$

Let us make the following change of variables. Denote

$$t = \tan(u/2).$$

Then

$$\begin{aligned} \cos u &= \cos 2(u/2) = \cos^2(u/2) - \sin^2(u/2) \\ &\left(\text{use the relation } \cos^2(u/2) + \sin^2(u/2) = 1 \right) \\ &= \frac{\cos^2(u/2) - \sin^2(u/2)}{\cos^2(u/2) + \sin^2(u/2)} \\ &\left(\text{divide numerator and denominator by } \cos^2(u/2) \right) \\ &= \frac{1 - \tan^2(u/2)}{1 + \tan^2(u/2)} = \frac{1 - t^2}{1 + t^2}. \end{aligned}$$

Moreover, $u = 2 \arctan t$, whence

$$du = (2 \arctan t)' \, dt = \frac{2dt}{1 + t^2}.$$

Substituting these in the integral we get

$$\begin{aligned} \int \sec u \, du &= \int \frac{du}{\cos u} = \int \frac{2dt}{1+t^2} \cdot \frac{1+t^2}{1-t^2} \\ &= \int \frac{2dt}{1-t^2} = \int \frac{2dt}{(1-t)(1+t)} = \int \frac{dt}{1-t} + \int \frac{dt}{1+t} \\ &= \ln|1-t| + \ln|1+t| + C = \ln|(1-t)(1+t)| + C = \ln|(1-t^2)| + C \\ &= \ln|(1 - \tan^2(u/2)| + C. \end{aligned}$$

Answer. $\int \sec u \, du = \ln|(1 - \tan^2(u/2)| + C$.