Answer on Question #50800 - Math - Calculus

Question

Show that if x is small, the $\sqrt{\frac{1+x}{1-x}}$ is approximated by $1+x+\frac{1}{2}x^2$. This sum is related to binomial expansion and there is a hint provided at the end of the question-that is,MULTIPLY BY $\sqrt{\frac{1+x}{1-x}}$ FIRST.

Solution

Let's consider function $f(x)=\sqrt{\frac{1+x}{1-x}}$ at the point $x=0$. Since $f(x)$ is more than 2 times differentiable at the point $x=0$ then due to the Taylor's theorem $f(x)-P_2(x)=o(|x|^2), x\to 0$, where $P_2(x)$ is the 2-th order Taylor polynomial.

$P_2(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2$.

Let's compute the first and the second order derivatives of $f(x)$:

$$f'(x)=\left(\sqrt{\frac{1+x}{1-x}}\right)'=\frac{1}{2}\sqrt{\frac{1-x}{1+x}}\frac{(1-x)+(1+x)}{(1-x)^2}=\sqrt{\frac{1-x}{1+x}}\frac{1}{(1-x)^2}=\sqrt{\frac{1}{(1+x)(1-x)^3}}=\left((1+x)(1-x)^3\right)^{-\frac{1}{2}};$$

$$f''(x)=\left(\left((1+x)(1-x)^3\right)^{-\frac{1}{2}}\right)'=-\frac{1}{2}\left((1+x)(1-x)^3\right)^{-\frac{3}{2}}\left((1-x)^3-3(1+x)(1-x)^2\right)$$

Thus, $P_2(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2=\sqrt{\frac{1+0}{1-0}}+\left((1+0)(1-0)^3\right)^{-\frac{1}{2}}x-$

$$-\frac{1}{2!}\frac{1}{2}\left((1+0)(1-0)^3\right)^{-\frac{3}{2}}\left((1-0)^3-3(1+0)(1-0)^2\right)x^2=1+x+\tfrac{1}{2}x^2.$$

As we say earlier, $f(x)-\left(1+x+\frac{1}{2}x^2\right)=o(|x|^2), x\to 0$, this means that $f(x)$ is approximated by $1+x+\frac{1}{2}x^2$ for the small x.