Answer on Question #50745 - Math – Calculus

Obtain the Fourier series for the following periodic function which has a period of 2π:

$$f(x)=\begin{cases}-\frac{1}{2}(\pi-x), & -\pi<x<0\\ \frac{1}{2}(\pi+x), & 0<x<\pi\end{cases}$$

Solution

Let's compute coefficients of the Fourier series:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}\left(\int_0^{\pi}(\pi+x)dx-\int_{-\pi}^{0}(\pi-x)dx\right)=\frac{1}{2\pi}\left((\pi x+\tfrac{1}{2}x^2)\Big|_0^{\pi}-(\pi x-\tfrac{1}{2}x^2)\Big|_{-\pi}^{0}\right)=$$

$$=\frac{1}{2\pi}\left(\tfrac{3}{2}\pi^2-\tfrac{3}{2}\pi^2\right)=0$$

for $n\ge 1$:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{2\pi}\left(\int_0^{\pi}(\pi+x)\cos nx\,dx-\int_{-\pi}^{0}(\pi-x)\cos nx\,dx\right)=$$

$$=\frac{1}{2\pi}\left(\int_0^{\pi}\pi\cos nx\,dx-\int_{-\pi}^{0}\pi\cos nx\,dx+\int_0^{\pi}x\cos nx\,dx+\int_{-\pi}^{0}x\cos nx\,dx\right)=$$

$$=\frac{1}{2\pi}\left(\frac{\pi}{n}\sin nx\Big|_0^{\pi}-\frac{\pi}{n}\sin nx\Big|_{-\pi}^{0}+\int_0^{\pi}x\cos nx\,dx+\int_{-\pi}^{0}x\cos nx\,dx\right)=\frac{1}{2\pi}\left(\int_0^{\pi}x\cos nx\,dx+\int_{-\pi}^{0}x\cos nx\,dx\right)$$

Due to integration by parts formula, where $u(x)=x$ and $dv(x)=\cos nx\,dx$, we obtain $du(x)=dx$ and $v(x)=\frac{1}{n}\sin nx$. Thus, $\int x\cos nx\,dx=\frac{1}{n}x\sin nx-\frac{1}{n}\int\sin nx\,dx=\frac{1}{n}x\sin nx+\frac{1}{n^2}\cos nx+C$, where C is an arbitrary real constant.

That is why

$$\frac{1}{2\pi}\left(\int_0^{\pi}x\cos nx\,dx+\int_{-\pi}^{0}x\cos nx\,dx\right)=\frac{1}{2\pi}\left((\tfrac{1}{n}x\sin nx+\tfrac{1}{n^2}\cos nx)\Big|_0^{\pi}+(\tfrac{1}{n}x\sin nx+\tfrac{1}{n^2}\cos nx)\Big|_{-\pi}^{0}\right)=$$

$$=\frac{1}{n^2 2\pi}\left((-1)^n-1+1-(-1)^n\right)=0$$

Thus, $a_n=0$, for $n\ge 1$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{2\pi}\left(\int_0^{\pi}(\pi+x)\sin nx\,dx-\int_{-\pi}^{0}(\pi-x)\sin nx\,dx\right)=$$

$$=\frac{1}{2\pi}\left(\int_0^{\pi}\pi\sin nx\,dx-\int_{-\pi}^{0}\pi\sin nx\,dx+\int_0^{\pi}x\sin nx\,dx+\int_{-\pi}^{0}x\sin nx\,dx\right)=\frac{1}{2n}\left(\cos nx\Big|_0^{\pi}-\cos nx\Big|_{-\pi}^{0}\right)+$$

$$+\frac{1}{2\pi}\left(\int_0^{\pi}x\sin nx\,dx+\int_{-\pi}^{0}x\sin nx\,dx\right)=\frac{1}{n}\left((-1)^n-1\right)+\frac{1}{2\pi}\left(\int_0^{\pi}x\sin nx\,dx+\int_{-\pi}^{0}x\sin nx\,dx\right)$$

Due to integration by parts formula, where $u(x)=x$ and $dv(x)=\sin nx\,dx$, we obtain $du(x)=dx$ and $v(x)=-\frac{1}{n}\cos nx$. Thus, $\int x\sin nx\,dx=-\frac{1}{n}x\cos nx+\frac{1}{n}\int\cos nx\,dx=-\frac{1}{n}x\cos nx+\frac{1}{n^2}\sin nx+C$, where C is an arbitrary real constant.

That is why

$$\frac{1}{2\pi}\left(\int_0^{\pi} x\sin nx dx + \int_{-\pi}^{0} x\sin nx dx\right) = \frac{1}{2\pi}\left((-\tfrac{1}{n}x\cos nx + \tfrac{1}{n^2}\sin nx)\Big|_0^{\pi} + (-\tfrac{1}{n}x\cos nx + \tfrac{1}{n^2}\sin nx)\Big|_{-\pi}^{0} \right) =$$

$$= \frac{1}{2n}\left(1-(-1)^n - 1 + (-1)^n\right) = 0 .$$

Thus, $b_n = \frac{1}{n}\left((-1)^n - 1\right)$, for $n \geq 1$

Therefore we obtain the Fourier series of the function $f(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{1}{n}\left((-1)^n - 1\right)\sin nx$$

Answer: $f(x) \sim \sum_{n=1}^{\infty} \frac{1}{n}\left((-1)^n - 1\right)\sin nx$