Answer on Question #50743 – Math – Differential Calculus | Equations

Solve the following ODE using the power series method:

(1 - X^2) y" - 2xy' + 2 y = 0

Solution

Ordinary differential equation

$$(1-x^2)y''-2xy'+2y=0$$

is equivalent to

$$y''-\frac{2x}{1-x^2}y'+\frac{2}{1-x^2}y=0.$$

Let

$$y=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+a_3x^3+\cdots,\quad y'=\sum_{n=1}^{\infty}na_nx^{n-1}\ ,\quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

Substitute $x=0$ **into** $y=\sum_{n=0}^{\infty}a_nx^n$**, which results in** $y(0)=a_0=0.$

Substitute $x=0$ **into** $y=\sum_{n=1}^{\infty}na_nx^{n-1}$**, which results in** $y'(0)=a_1=1.$

We are searching for two linearly independent solutions of differential equation, so let the first one be such that $y(0)=0, y'(0)=1$**, hence** $a_0=0$**,** $a_1=1$

Substitute $y=\sum_{n=0}^{\infty}a_nx^n,\quad y'=\sum_{n=1}^{\infty}na_nx^{n-1}\ ,\ y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$

into initial differential equation $(1-x^2)y''-2xy'+2y=0$**, which results in**

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^n-2\sum_{n=1}^{\infty}na_nx^n+2\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty}[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n]x^n + 2a_2 + 6a_3 x - 2a_1 x + 2a_0 + 2a_1 x = 0$$

$$\sum_{n=2}^{\infty}[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n]x^n + 2a_2 + 6a_3 x + 2a_0 = 0$$

Plug $x = 0$ and $a_0 = 0$ into the last equation, obtain $a_2 = 0$. Rewrite the series as

$$\sum_{n=2}^{\infty}[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n]x^n + 6a_3 x = 0$$

Divide both sides by $x \neq 0$ and obtain

$$\sum_{n=2}^{\infty}[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n]x^{n-1} + 6a_3 = 0$$

Plug $x = 0$ into the last equation, obtain $6a_3 = 0$, hence $a_3 = 0$.

Thus we have $a_1 = 1,\ a_0 = a_2 = a_3 = 0$.

Rewrite the series as

$\sum_{n=2}^{\infty}[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n]x^{n-1} = 0.$

A power series is identically equal to zero if and only if all of its coefficients are equal to zero.

Hence

$$(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n = 0 \ \rightarrow$$

$$\rightarrow \quad a_{n+2} = \frac{n^2+n-2}{(n+1)(n+2)}a_n = \frac{n-1}{n+1}a_n$$

Read the recurrence relation for the case $n = 0$: $a_2 = -a_0$. Reading off the recurrence relation for $n = 1$ yields $a_3 = 0$.

Notice that because $a_2 = a_3 = 0$, all rest coefficients must be zero. Therefore the first linear independent solution is $y = x$.

Points $x = 1$ and $x = -1$ are singular points of equation

$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0.$$

The first linearly independent solution is found.

To obtain the second linearly independent solution, the next formula is used:

$$F = \int \frac{f_1}{f_2}dx = \int \frac{-2x}{1-x^2}dx = \int \frac{d(1-x^2)}{1-x^2} = ln\left|1-x^2\right|,$$

general solution can be represented as

$y = y_1\left(C_1 + C_2\int \frac{e^{-F}}{y_1^2}dx\right) = x\left(C_1 + C_2\int \frac{e^{-ln\left|1-x^2\right|}}{x^2}dx\right)$**, where C_1 and C_2 are arbitrary real constants.**

Let $-1 < x < 1$, then

$$y = y_1\left(C_1 + C_2\int \frac{e^{-F}}{y_1^2}dx\right) = x\left(C_1 + C_2\int \frac{e^{-ln\left(1-x^2\right)}}{x^2}dx\right) =$$

$$= x\left(C_1 + C_2\int \frac{dx}{(1-x^2)x^2}\right) = x\left(C_1 - C_2\int \frac{dx}{(x^2-1)x^2}\right)$$

$$= x\left(C_1 - C_2\int\left(\frac{1}{x^2-1} - \frac{1}{x^2}\right)dx\right)$$

$$= x\left(C_1 - C_2\left(\frac{1}{2}ln\left|\frac{x-1}{x+1}\right| + \frac{1}{x}\right)\right)$$

$$= x\left(C_1 - C_2\left(ln\sqrt{\left|\frac{x-1}{x+1}\right|} + \frac{1}{x}\right)\right)$$

Other method is to search for the second linearly independent solution in the form

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$.