Solve the following ordinary differential equations:

I)

$$(2y+x^2+1)\frac{dy}{dx}+2xy-9x^2=0;$$

2)

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=x^2.$$

Solution:

1)

Denote

$$M(x,y)=2xy-9x^2, N(x,y)=2y+x^2+1.$$

Because

$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}$$

then the differential equation

$$(2y+x^2+1)\frac{dy}{dx}+2xy-9x^2=0$$

has general solution

$$U(x,y)=c=const$$

where

$$\frac{\partial U}{\partial x}=M(x,y)=2xy-9x^2,$$

$$\frac{\partial U}{\partial y}=N(x,y)=2y+x^2+1.$$

From first equation we have

$$U(x,y)=\int M(x,y)dx=\int(2xy-9x^2)dx=yx^2-3x^3+\varphi(y).$$

Thus

$$\frac{\partial}{\partial y}\left(yx^2-3x^3+\varphi(y)\right)=N(x,y)=2y+x^2+1,$$

$$x^2 + \frac{d\varphi}{dy} = 2y + x^2 + 1,$$

$$\frac{d\varphi}{dy} = 2y + 1,$$

$$d\varphi = (2y + 1)dy,$$

$$\int d\varphi = \int (2y + 1)dy,$$

$$\varphi(y) = y^2 + y.$$

Thus the general solution is

$$\boxed{yx^2 - 3x^3 + y^2 + y - c = 0}$$

2)

The characteristic equation is

$$k^2 + 3k + 2 = 0,$$

$$k_1 = -1, k_2 = -2.$$

So general solution of the equation is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + u(x),$$

where

$$u(x) = ax^2 + bx + c,\ \ c_1 = const,\ c_2 = const, a = const, b = const, c = const.$$

Substitution $u(x)$ into the equation we have

$$u''(x) + 3u'(x) + 2u(x) = x^2,$$

$$2a + 3(2ax + b) + 2\left(ax^2 + bx + c\right) = x^2,$$

$$2ax^2 + (6a + 2b)x + 2a + 3b + 2c = x^2.$$

Thus

$$\begin{cases} 2a = 1, \\ 6a + 2b = 0, \\ 2a + 3b + 2c = 0, \end{cases} \rightarrow \begin{cases} a = \frac{1}{2}, \\ b = -1\frac{1}{2}, \\ c = 1\frac{3}{4}. \end{cases}$$

So

$$u(x) = \frac{1}{2}x^2 - 1\frac{1}{2}x + 1\frac{3}{4}$$

and finally

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}x^2 - 1\frac{1}{2}x + 1\frac{3}{4}}.$$

Answers:

1)

$$\boxed{yx^2 - 3x^3 + y^2 + y - c = 0}$$

2)

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}x^2 - 1\frac{1}{2}x + 1\frac{3}{4}}$$