**Answer on Question #50729 – Math – Linear Algebra**

Show the matrix [r1= -9 4 4, r2= -8 3 4, r3= -16 8 7,] of order 3*3 is diagonalizable. Obtain the diagonalizable matrix P.

**Solution**

We have

$$A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}.$$

Compose the characteristic equation

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0 \rightarrow |subtract\ the\ first\ row\ from\ the\ second\ one|$$

$$\rightarrow \begin{vmatrix} -9-\lambda & 4 & 4 \\ \lambda+1 & -\lambda-1 & 0 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0 \rightarrow |factor\ out\ (\lambda+1)\ from\ the\ second\ row|$$

$$\rightarrow (\lambda+1)\begin{vmatrix} -9-\lambda & 4 & 4 \\ 1 & -1 & 0 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0 \rightarrow |add\ the\ first\ column\ to\ the\ second\ one|$$

$$\rightarrow (\lambda+1)\begin{vmatrix} -9-\lambda & -\lambda-5 & 4 \\ 1 & 0 & 0 \\ -16 & -8 & 7-\lambda \end{vmatrix} = 0 \rightarrow |expand\ along\ the\ second\ row|$$

$$\rightarrow -(\lambda+1)\begin{vmatrix} -\lambda-5 & 4 \\ -8 & 7-\lambda \end{vmatrix} = 0 \rightarrow -(\lambda+1)\big((\lambda+5)(\lambda-7)+32\big) = 0$$

$$\rightarrow (\lambda+1)(\lambda^2-2\lambda-35+32) = 0 \rightarrow (\lambda+1)(\lambda^2-2\lambda-3) = 0$$

$$\rightarrow (\lambda+1)(\lambda^2-2\lambda+1-4) = 0 \rightarrow (\lambda+1)(\lambda-1-2)(\lambda-1+2) = 0 \rightarrow$$

$(\lambda-3)(\lambda+1)^2 = 0$, hence $\lambda_1 = 3, \lambda_2 = -1, \lambda_3 = -1.$

If $\lambda_1 = 3$, then

$$\begin{pmatrix} -9-3 & 4 & 4 \\ -8 & 3-3 & 4 \\ -16 & 8 & 7-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow -8x+4z = 0 \rightarrow z = 2x;$$

$$-12x+4y+4z = 0 \rightarrow -12x+4y+8x = 0 \rightarrow x = y.$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, where $C \neq 0$.

If $\lambda_2 = -1$ or $\lambda_3 = -1$, then

$$\begin{pmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow y+z = 2x.$$

There are 3 variables and 1 equation.

If $y = 2, z = 0$, then $x = 1$. If $y = 0, z = 2$, then $x = 1$.

Therefore

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, where vectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $and$ $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ are linearly independent in $\mathbb{R}^3$.

Thus, matrix $A$ is diagonalizable,

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}.$$