Answer on Question# #50671 – Math– Differential Calculus | Equations

Question

Find the solution of the Riccati equation: dy/dx = 2cos^2 x - sin^2 x + y^2/ 2cosx ; y1(x) = sinx.

Note that there is an error in the form of given equation! The function $y_1(x) = sin(x)$ is not a particular solution of equation

$$\frac{dy}{dx} = 2cos^2(x) - sin^2(x) + \frac{y^2}{2\cos(x)},$$

$$\left(\frac{dy_1}{dx} \neq 2cos^2(x) - sin^2(x) + \frac{{y_1}^2}{2\cos(x)}\right)$$

but it satisfies the equation

$$\frac{dy}{dx} = \frac{2cos^2(x) - sin^2(x)}{2\cos(x)} + \frac{1}{2\cos(x)} y^2, \qquad (*)$$

$$\left(\frac{dy_1}{dx} = \frac{2cos^2(x) - sin^2(x)}{2\cos(x)} + \frac{1}{2\cos(x)} {y_1}^2\right).$$

So we will solve the equation (*).

Solution

The Riccati equation is the first-order nonlinear ordinary differential equation. In the general case it is an equation of the type

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x) \qquad (1)$$

where *P(x), Q(x), R(x)* are continuous functions. In our case

$$P(x) = \frac{1}{2\cos(x)}, \qquad Q(x) = 0, \qquad R(x) = \frac{2cos^2(x) - sin^2(x)}{2\cos(x)}$$

and equation (1) takes the form

$$\frac{dy}{dx} = \frac{1}{2\cos(x)} y^2 + \frac{2cos^2(x) - sin^2(x)}{2\cos(x)}. \qquad (2)$$

If we know a particular solution of the Riccati equation, then its general solution is

$$y(x) = y_1(x) + u(x), \qquad (3)$$

where *u(x)* is a new unknown function. In our case the particular solution is $y_1(x) = sin(x)$.

Let us solve equation (2) knowing the particular solution $y_1(x)$.

Substituting (3) into (2) we have

$$\frac{d}{dx}(y_1+u)=\frac{1}{2\cos(x)}(y_1+u)^2+\frac{2cos^2(x)-sin^2(x)}{2\cos(x)}$$

$$\frac{dy_1}{dx}+\frac{du}{dx}=\frac{1}{2\cos(x)}(y_1)^2+\frac{2y_1u}{2\cos(x)}+\frac{u^2}{2\cos(x)}+\frac{2cos^2(x)-sin^2(x)}{2\cos(x)} \qquad (4)$$

As $y_1(x)$ *is a* function that satisfies the equation (2), then

$$\frac{dy_1}{dx}=\frac{1}{2\cos(x)}{y_1}^2+\frac{2cos^2(x)-sin^2(x)}{2\cos(x)} \qquad (5)$$

Thus, from (4), taking into account (5), we obtain the differential equation for the function *u(x)*

$$\frac{du}{dx}=\frac{2y_1u}{2\cos(x)}+\frac{u^2}{2\cos(x)} \qquad (6)$$

It is the Bernoulli equation. The substitution

$$z(x)=\frac{1}{u(x)}. \qquad (7)$$

transforms the equation (6) into a linear differential equation admitting integration:

$$\frac{dz}{dx}+ztan(x)=-\frac{1}{2\cos(x)}. \qquad (8)$$

Further, to find the solution of equation (8) we use the variation of constants method:

$$\frac{dz}{dx}+ztan(x)=0 \Longrightarrow$$

$$\frac{dz}{z}+\frac{\sin(x)}{\cos(x)}dx=0 \Longrightarrow$$

$$ln|z|-ln|\cos(x)|=ln|C| \Longrightarrow$$

$$z(x)=C(x)\cos(x).$$

$$\frac{d}{dx}(C(x)\cos(x))+(C(x)\cos(x))tan(x)=-\frac{1}{2\cos(x)} \Longrightarrow$$

$$(C'(x)\cos(x)-C(x)\sin(x))+(C(x)\cos(x))\frac{\sin(x)}{\cos(x)}=-\frac{1}{2\cos(x)} \Longrightarrow$$

$$C'(x)\cos(x)-C(x)\sin(x)+C(x)\sin(\mathrm{x})=-\frac{1}{2\cos(x)} \Longrightarrow$$

$$C'(\mathrm{x})\cos(\mathrm{x})=-\frac{1}{2\cos(x)} \Longrightarrow$$

$$C'(\mathrm{x})=-\frac{1}{2(\cos(x))^2} \Longrightarrow$$

$$C(x)=-\int\frac{dx}{2(\cos(x))^2}=-\frac{\tan(x)}{2}+C_1 \Longrightarrow$$

$$z(x) = C(x)\cos(x) = \left(-\frac{\tan(x)}{2} + C_1\right)\cos(x) = C_1\cos(x) - \frac{1}{2}\sin(x).$$

$$z(x) = C_1\cos(x) - \frac{1}{2}\sin(x) = \frac{1}{2}\left(\widetilde{C_1}\cos(x) - \sin(x)\right), \quad \widetilde{C_1} = 2C_1,$$

$$z(x) = \frac{1}{2}\left(\widetilde{C_1}\cos(x) - \sin(x)\right). \tag{9}$$

Hence, using (7) and (3) we get solution of equation (2):

$$y(x) = \sin(x) + \frac{2}{\widetilde{C_1}\cos(x) - \sin(x)}, \tag{10}$$

where $\widetilde{C_1}$ is an arbitrary real constant.

Let us check whether the function *y(x)* satisfies the equation (2):

$$\frac{d}{dx}\left(\sin(x) + \frac{2}{\widetilde{C_1}\cos(x)-\sin(x)}\right) = \frac{1}{2\cos(x)}\left(\sin(x) + \frac{2}{\widetilde{C_1}\cos(x)-\sin(x)}\right)^2 + \frac{2cos^2(x)-sin^2(x)}{2\cos(x)},$$

$$\left(\cos(x) + \frac{2(\widetilde{C_1}\sin(x)+\cos(x))}{(\widetilde{C_1}\cos(x)-\sin(x))^2}\right) = \frac{1}{2\cos(x)}\left(\sin^2(x) + \frac{4\sin(x)}{\widetilde{C_1}\cos(x)-\sin(x)} + \frac{4}{(\widetilde{C_1}\cos(x)-\sin(x))^2}\right) + \frac{2cos^2(x)-sin^2(x)}{2\cos(x)},$$

$$\left(\cos(x) + \frac{2(\widetilde{C_1}\sin(x)+\cos(x))}{(\widetilde{C_1}\cos(x)-\sin(x))^2}\right) = \frac{\sin^2(x)}{2\cos(x)} + \frac{4\sin(x)}{2\cos(x)(\widetilde{C_1}\cos(x)-\sin(x))} + \frac{4}{2\cos(x)(\widetilde{C_1}\cos(x)-\sin(x))^2} + \frac{2cos^2(x)-sin^2(x)}{2\cos(x)},$$

$$\frac{2(\widetilde{C_1}\sin(x)+\cos(x))}{(\widetilde{C_1}\cos(x)-\sin(x))^2} = \frac{4\sin(x)}{2\cos(x)(\widetilde{C_1}\cos(x)-\sin(x))} + \frac{4}{2\cos(x)(\widetilde{C_1}\cos(x)-\sin(x))^2},$$

$$\frac{2(\widetilde{C_1}\sin(x)+\cos(x))}{(\widetilde{C_1}\cos(x)-\sin(x))^2} = \frac{2}{\cos(x)}\left(\frac{\sin(x)(\widetilde{C_1}\cos(x)-\sin(x))+1}{(\widetilde{C_1}\cos(x)-\sin(x))^2}\right),$$

$$\frac{2(\widetilde{C_1}\sin(x)+\cos(x))}{(\widetilde{C_1}\cos(x)-\sin(x))^2} = \frac{2}{\cos(x)}\left(\frac{\widetilde{C_1}\sin(x)\cos(x)+cos^2(x)}{(\widetilde{C_1}\cos(x)-\sin(x))^2}\right),$$

$$\frac{2(\widetilde{C_1}\sin(x)+\cos(x))}{(\widetilde{C_1}\cos(x)-\sin(x))^2} = \frac{2(\widetilde{C_1}\sin(x)+\cos(x))}{(\widetilde{C_1}\cos(x)-\sin(x))^2}.$$

As we see, the function (10) satisfies the equation (2).

Answer: $y(x) = \sin(x) + \frac{2}{\widetilde{C_1}\cos(x)-\sin(x)}$.