Answer on Question #50669 - Math - Differential calculus | Equations

Question

Find the differential equation of the family of curves $y = e^x(A\cos x + B\sin x)$ where A and B are arbitrary constants and hence solve the equation.

Solution

Method 1

From the theory of ordinary differential equations the following fact is known:
if $y = e^x(A\cos x + B\sin x)$ is the solution to differential equation, then the characteristic equation has complex roots $\lambda_{1,2} = 1 \pm i$. We obtain characteristic equation of the original differential equation: $(\lambda - 1 - i)(\lambda - 1 + i) = 0 \Leftrightarrow (\lambda - 1)^2 = -1 \Leftrightarrow \lambda^2 - 2\lambda + 2 = 0$

Therefore the original differential equation is $y'' - 2y' + 2y = 0$.

If we want to solve it, then we must repeat all previous steps.

Let's construct characteristic equation of the original differential equation: $\lambda^2 - 2\lambda + 2 = 0$. It is easy to see that solutions of the characteristic equation are $\lambda_{1,2} = 1 \pm i$. Since $\lambda_{1,2} = 1 \pm i$ are complex numbers with real part 1 and image part ± 1 then the family of curves $y = e^x(A\cos x + B\sin x)$, where A and B are arbitrary real constants, is solution of differential equation $y'' - 2y' + 2y = 0$.

Method 2

To find the differential equation of the family of curves, differentiate $y = e^x(A\cos x + B\sin x)$:

$$y' = e^x(A\cos x + B\sin x - A\sin x + B\cos x = e^x\cos x(A + B) + e^x\sin x(B - A)$$

$$y'' = e^x(A\cos x + B\sin x - 2A\sin x + 2B\cos x - A\cos x - B\sin x) = 2Be^x\cos x - 2Ae^x\sin x$$

Suppose that the family of curves $y = e^x(A\cos x + B\sin x)$ satisfies equation $y' + \mu y = 0$, then $\frac{A + B}{A} = \frac{B - A}{B}$, which gives infinitely many cases $A^2 + B^2 = 1$, because A and B are arbitrary constants, so we failed to construct the first-order linear differential equation with constant coefficients.

Solve the equation $y'' + \alpha y' + \beta y = 0$ for α, β:

$$e^x\cos x(2B + \alpha(A + B) + \beta A) + e^x\sin x(-2A + \alpha(B - A) + \beta B) = 0$$

for arbitrary real x, hence obtain the system of equations

$$\begin{cases} (A + B)\alpha + A\beta = -2B \\ (B - A)\alpha + B\beta = 2A \end{cases}$$

$$\Delta = \begin{vmatrix} A + B & A \\ B - A & B \end{vmatrix} = AB + B^2 - AB + A^2 = A^2 + B^2,\ \Delta_1 = \begin{vmatrix} -2B & A \\ 2A & B \end{vmatrix} = -2B^2 - 2A^2 = -2(A^2 + B^2),$$

$$\Delta_2 = \begin{vmatrix} A + B & -2B \\ B - A & 2A \end{vmatrix} = 2A^2 + 2AB + 2B^2 - 2AB = 2(A^2 + B^2).$$

By Cramer's rule, obtain the unique solution $\alpha = \frac{\Delta_1}{\Delta} = \frac{-2(A^2 + B^2)}{A^2 + B^2} = -2$, $\beta = \frac{\Delta_2}{\Delta} = \frac{2(A^2 + B^2)}{A^2 + B^2} = 2$.

Finally the differential equation will be $y'' - 2y' + 2y = 0$.

Answer: $y'' - 2y' + 2y = 0$.