1. Using the definition of improper integrals, evaluate the integral ,or show that it diverges.

$$\int_c^1 \frac{1}{\sqrt[3]{x}}dx$$

Solution.

1) If $c > 0$, then the integral is proper.

$$\int_c^1 \frac{dx}{\sqrt[3]{x}} = \frac{3}{2}x^{2/3}\Big|_c^1 = \frac{3}{2}\left(1 - c^{2/3}\right).$$

2) If $c = 0$, then $\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{\varepsilon\to+0}\int_\varepsilon^1 \frac{dx}{\sqrt[3]{x}} = \lim_{\varepsilon\to+0}\frac{3}{2}x^{2/3}\Big|_\varepsilon^1 = \lim_{\varepsilon\to+0}\frac{3}{2}\left(1-\varepsilon^{2/3}\right) = \frac{3}{2}\left(1-0^{2/3}\right) = \frac{3}{2}.$

3) If $c < 0$, then $\int_c^1 \frac{dx}{\sqrt[3]{x}} = \lim_{\varepsilon\to+0}\left(\int_c^{-\varepsilon}\frac{dx}{\sqrt[3]{x}} + \int_\varepsilon^1 \frac{dx}{\sqrt[3]{x}}\right) = \left|t = -x\right| = \lim_{\varepsilon\to+0}\left(\int_c^{\varepsilon}\frac{dt}{\sqrt[3]{t}} + \int_\varepsilon^1 \frac{dx}{\sqrt[3]{x}}\right) =$

$$= \lim_{\varepsilon\to+0}\left[\frac{3}{2}t^{2/3}\Big|_{-c}^{\varepsilon} + \frac{3}{2}x^{2/3}\Big|_\varepsilon^1\right] = \frac{3}{2}\lim_{\varepsilon\to+0}\left[\varepsilon^{2/3} - (-c)^{2/3} + 1 - \varepsilon^{2/3}\right] = \frac{3}{2}\lim_{\varepsilon\to+0}\left[1-(-c)^{2/3}\right] = \frac{3}{2}\left[1-(-c)^{2/3}\right].$$

Answer: $\frac{3}{2}\left(1-c^{2/3}\right)$, if $c > 0$; $\frac{3}{2}\left[1-(-c)^{2/3}\right]$, if $c \le 0$.