Answer on Question#50256 – Math – Other

$$\pi\left(1-\frac{2\sqrt{14}}{7}\right)$$

## Solution

$$\int_0^{2\pi}\frac{\sin\theta}{4+\sin\theta+\cos\theta}\,d\theta$$

General formula:

$$\int_0^{2\pi}R(\cos\theta,\sin\theta)\,d\theta=\oint_{U(0,1)}R\left(\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{dz}{iz}$$

, where $U(0,1)$ – unit circle centered at origin.

$$\int_0^{2\pi}\frac{\sin\theta}{4+\sin\theta+\cos\theta}\,d\theta=\oint_{U(0,1)}\frac{\frac{1}{2i}\left(z-\frac{1}{z}\right)}{4+\frac{1}{2i}\left(z-\frac{1}{z}\right)+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}=$$

$$\oint_{U(0,1)}\frac{1-z^2}{z\big((1-i)z^2+8z+(1+i)\big)}dz=$$

$$2\pi i\sum\operatorname*{Res}_{U(0,1)}\frac{1-z^2}{z\big((1-i)z^2+8z+(1+i)\big)}$$

, summation over all singularities within unit circle.

Let us determine singularities.

Examine zeros of numerator.

$$1-z^2=0$$

$$z^2=1$$

$$z_{n1}=1$$

$$z_{n2}=-1$$

Examine zeros of denominator.

$$z=0$$

$$z_1=0-simple\ pole$$

$$(1-i)z^2+8z+(1+i)=0$$

$$z_2=\left(-2-\sqrt{\frac{7}{2}}\right)(1+i)-simple\ pole$$

$$z_3 = \left(-2 + \sqrt{\frac{7}{2}}\right)(1+i) - simple\ pole$$

Look, which singularities are inside the unit circle.

$$z_1 = 0 - inside$$

$$z_2 = \left(-2 - \sqrt{\frac{7}{2}}\right)(1+i) \approx -3.9 + 3.9i - outside$$

$$z_3 = \left(-2 + \sqrt{\frac{7}{2}}\right)(1+i) \approx -0.13 + 0.13i - inside$$

Calculate related residues.

As soon as we work with simple poles, we can use formula:

$$\operatorname*{Res}_{z=a} \frac{\phi(z)}{\psi(z)} = \frac{\phi(a)}{\psi'(a)}$$

, where $\phi(a) \neq 0; \psi(a) = 0; \psi'(a) \neq 0.$

To use it effectively rearrange expression.

$$\frac{1-z^2}{z\big((1-i)z^2 + 8z + (1+i)\big)} = \frac{\left(\frac{1+i}{2}\right) - \left(\frac{1+i}{2}\right)z^2}{z(z^2 + 4(1+i)z + i)} = \frac{A}{z} + \frac{Bz + C}{z^2 + 4(1+i)z + i}$$

$z^0$: $iA = \frac{1+i}{2} \rightarrow A = \frac{1-i}{2}$

$z^1$: $4(1+i)A + C = 0 \rightarrow C = -4A(1+i) = -\frac{4}{2}(1+i)(1-i) = -4$

$z^2$: $A + B = -\left(\frac{1+i}{2}\right) \rightarrow B = -\left(\frac{1+i}{2}\right) - A = -\left(\frac{1+i}{2}\right) - \left(\frac{1-i}{2}\right) = -1$

$$\frac{1-z^2}{z\big((1-i)z^2 + 8z + (1+i)\big)} = \frac{\frac{1-i}{2}}{z} + \frac{-z-4}{z^2 + 4(1+i)z + i}$$

Recall that we shall use formula only to addenda, which contain singularity.

$$\operatorname*{Res}_{z=0} \frac{\frac{1-i}{2}}{z} = \frac{1-i}{2}$$

$$\operatorname*{Res}_{z=z_3} \frac{-z-4}{z^2 + 4(1+i)z + i} = \frac{-z_3 - 4}{2z_3 + 4(1+i)} =$$

$$\frac{-\left(-2 + \sqrt{\frac{7}{2}}\right)(1+i) - 4}{2\left(-2 + \sqrt{\frac{7}{2}}\right)(1+i) + 4(1+i)} =$$

$$\frac{2+2i-\sqrt{\frac{7}{2}}-i\sqrt{\frac{7}{2}}-4}{-4-4i+\sqrt{14}+i\sqrt{14}+4+4i}=\frac{-\left(2+\sqrt{\frac{7}{2}}\right)+i\left(2-\sqrt{\frac{7}{2}}\right)}{\sqrt{14}(1+i)}=$$

$$\frac{-\left(2+\sqrt{\frac{7}{2}}\right)+i\left(2-\sqrt{\frac{7}{2}}\right)+i\left(2+\sqrt{\frac{7}{2}}\right)+\left(2-\sqrt{\frac{7}{2}}\right)}{2\sqrt{14}}=$$

$$\frac{-2\sqrt{\frac{7}{2}}+4i}{2\sqrt{14}}=\frac{-7+2\sqrt{14}i}{14}=-\frac{1}{2}+\frac{\sqrt{14}}{7}i$$

Finally:

$$2\pi i\sum \operatorname*{Res}_{U(0,1)}\frac{1-z^2}{z\big((1-i)z^2+8z+(1+i)\big)}=2\pi i\left(\operatorname*{Res}_{z=0}\frac{\frac{1-i}{2}}{z}+\operatorname*{Res}_{z=z_3}\frac{-z-4}{z^2+4(1+i)z+i}\right)=$$

$$2\pi i\left(\left(\frac{1-i}{2}\right)+\left(-\frac{1}{2}+\frac{\sqrt{14}}{7}i\right)\right)=2\pi i\left(-\frac{i}{2}+\frac{\sqrt{14}}{7}i\right)=\pi\left(1-\frac{2\sqrt{14}}{7}\right)$$