Answer on Question #50235 – Math - Complex Analysis

Find the value(s) of constant B such that :
integral on curve for [(1)- (3z) + (2 B {z^4}) +(z^6) + (3 {z^7}) + (11 {z ^ 100}] /[z^5] dz =
integral on curve for [e ^ {Bz} + 2 z] / [z^3] dz
where C is the unit circle oriented counterclockwise .

Solution

$$I_1 = \oint f_1(z)\,dz = I_2 = \oint f_2(z)dz$$

where $f_1(z) = \dfrac{1-3z+2Bz^4+z^6+3z^7+11z^{100}}{z^5}$, $f_2(z) = \dfrac{e^{Bz}+2z}{z^3}$.

Both integrands have singularities at $z=0$ and the value of each integral can be evaluated as $I_1 = 2\pi\imath \cdot res[f_1(z), z=0] = I_2 = 2\pi\imath \cdot res[f_2(z), z=0]$

We can break up the integrand $f_1(z)$ into six fractions dividing each term in the numerator by the denominator $f_1(z) = \dfrac{1-3z+2Bz^4+z^6+3z^7+11z^{100}}{z^5} = \dfrac{1}{z^5} - 3\dfrac{1}{z^4} + 2B\dfrac{1}{z} + z + 3z^2 + 11z^{95}$, we see that $f_1(z)$ has a Laurent expansion about $z=0$ given by

$f_1(z) = \dfrac{1}{z^5} - 3\dfrac{1}{z^4} + 2B\dfrac{1}{z} + z + 3z^2 + 11z^{95}$. Hence the residue is $2B$ (the coefficient of z^{-1}).

$$\oint \frac{1-3z+2Bz^4+z^6+3z^7+11z^{100}}{z^5}dz = 2\pi\imath \cdot res[f_1(z), z=0] = 2\pi\imath\, c_{-1} = 2\pi\imath\, 2B$$

c_{-1}-is the coefficient of the Laurent expansion about $z=0$.

In a similar way by expanding $\dfrac{e^{Bz}+2z}{z^3}$ as a Taylor series, we see that

$f_2(z) = \dfrac{e^{Bz}+2z}{z^3} = \dfrac{1+Bz+B^2z^2/2+B^3z^3/6+2z}{z^3}$ has a Laurent expansion about $z=0$ given by $\dfrac{1}{z^3} + \dfrac{B+2}{z^2} + \dfrac{B^2}{2z} + B^3 + ...$

So, the residue is $\dfrac{B^2}{2}$ (the coefficient of z^{-1}). Hence the both integrals are equal if the next condition is fulfilled: $B^2/2 = 2B \Rightarrow \begin{cases} B=0 \\ B=4 \end{cases}$.

Answer: $\begin{cases} B=0 \\ B=4 \end{cases}$.