Answer on Question #50231- Math – Complex Analysis

Let f(z) be an analytic function in the annulus 0 <|z| < R for some positive real number R,Whose laurent series (in this annulus) is given by

f(z) = n from -∞ to ∞ ∑ { (-1)^n / (n^2)!] } . Z ^ { 5n - n^2 -1}
A)) What Kind of Singularity is z=0 for f(z) ?
B)) Compute integral on Curve for [z ^ 24 . f(z) dz] ,where C is a counterclockwise simple path lying in the annulus enclosing z=0
C)) Calculate Res (f) in z=0
D)) Evaluate Integral on Curve for [sin Z .f(z) dz] , where C : |z| = (R/2) oriented counterclockwise

Solution

A) The function

$$f(z) = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(n^2)!} z^{5n-n^2-1} = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(n^2)!} \frac{1}{z^{n^2-5n+1}} \quad (1)$$

has an singularity at point $z_0 = 0$, because it is not defined there, but it is defined at other points of the annulus 0 <|z| < R.

The function $f(z)$ has an essential singularity at 0, because according to (1), $f(z)$ contains infinitely many terms with negative powers of z ($5n - n^2 - 1 < -1$ for $n < 0$ and $n > 5$, i.e. there exist infinitely many terms with negative power of z).

B) The function

$$z^{24} f(z) = z^{24} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(n^2)!} z^{5n-n^2-1} = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(n^2)!} \frac{1}{z^{n^2-5n-24+1}} \quad (2)$$

has an essential singularity at point $z_0 = 0$.

We used $z^{24} z^{5n-n^2-1} = z^{24+5n-n^2-1} = z^{-(n^2-5n-24+1)} = \frac{1}{z^{n^2-5n-24+1}}$ according to properties of exponents and power functions, operations with real numbers

Coefficient c_{-1} of the term with $z^{-1} = \frac{1}{z} = \frac{1}{z^1}$ in (2) is called the residue of function $h(z) = z^{24} f(z)$ at point $z_0 = 0$, here point $z_0 = 0$ is finite.

(notation is the following: $Res_{z=0}(h) = Res_{z=0}(z^{24} f) = c_{-1}$).

In order to find the coefficient that correspond to values of n such that

$\frac{1}{z^{n^2-5n-24+1}} = \frac{1}{z^1}$, search for natural solutions to equation

$$n^2 - 5n - 24 + 1 = 1,$$

which is equivalent to quadratic equation

$$n^2 - 5n - 24 = 0,$$

its discriminant is

$$D = (-5)^2 - 4 \cdot (-24) = 25 + 4 \cdot 24 = 25 + 96 = 121,$$

hence $n = \frac{5 \pm 11}{2} = \frac{5-11}{2}; \frac{5+11}{2} = \frac{-6}{2}; \frac{16}{2} = -3; 8$

which implies that we take only natural solution $n = 8$, therefore, the coefficient c_{-1} of the term of (2) with z^{-1} will be $\frac{(-1)^n}{(n^2)!} = \frac{(-1)^8}{(8^2)!} = \frac{1}{64!}$, which leads to

$$Res_{z=0}(h) = Res_{z=0}(z^{24} f) = c_{-1} = \frac{1}{64!}.$$

Coefficient c_{-1} can be computed as $c_{-1} = \frac{1}{2\pi i} \oint_T \frac{h(z)}{(z-0)^{-1+1}} = \frac{1}{2\pi i} \oint_T h(z)dz$, here T is a path lying in the annulus ($0 < |z| < R$) enclosing $z=0$.

Thus, by residue theorem, $\oint_T h(z)dz = 2\pi i c_{-1} = 2\pi i \cdot \frac{1}{64!} = \frac{2\pi i}{64!}$ (we only deal with singularity 0).

C) Coefficient a_{-1} of the term with $z^{-1} = \frac{1}{z} = \frac{1}{z^1}$ in (1) is called the residue of function $f(z)$ at point $z_0 = 0$, here point $z_0 = 0$ is finite.

(notation is the following: $Res_{z=0}(f) = Res_{z=0}(f) = a_{-1}$).

In order to find the coefficient that correspond to values of n such that $\frac{1}{z^{n^2-5n+1}} = \frac{1}{z^1}$, search for natural solutions to equation

$$n^2 - 5n = 0 \Longrightarrow$$

$$n(n-5) = 0 \Longrightarrow n = 0, n = 5,$$

therefore, coefficient a_{-1} equals

$$\frac{(-1)^0}{(0^2)!} + \frac{(-1)^5}{(5^2)!} = 1 - \frac{1}{25!}.$$

Point $z_0 = 0$ is finite, hence residue $Res_{z=0} f = a_{-1} = 1 - \frac{1}{25!}$

D) Consider

$$g(z) = sinz \cdot f(z) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots\right) \times$$

$$\times \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(n^2)!} z^{5n-n^2-1} = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots\right) \times$$

$$\times \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(n^2)!} \frac{1}{z^{n^2-5n+1}} \qquad (3)$$

(here we sum up terms $\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{(n^2)!}\frac{1}{z^{n^2-5n+1}}$, multiplied by terms of
$z-\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)!}+\cdots$, which lead to the sum of terms with $\frac{z^{2k+1}}{z^{n^2-5n+1}}=\frac{1}{z^{n^2-5n+1-(2k+1)}}$ (according to properties of exponents and power functions), k is an index related to terms of the series $z-\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$, n is an index related to terms of the series $\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{(n^2)!}z^{5n-n^2-1}$, k is a non-negative integer).

In case of $g(z)=sinz\cdot f(z)$, k is integer, equate

$n^2-5n-(2k+1)+1=1\Longrightarrow$

$$n^2-5n=2k+1 \qquad (4)$$

If $n=2l$, l is integer, then equation (4) does not have natural solutions, because the left-hand side (i.e. $n^2-5n=4l^2-10l$) is even, but the right-hand side $(i.e. 2k+1)$ is odd .

If $n=2l+1,\ l$ is integer, then

$$(2l+1)^2-5(2l+1)=2k+1\Longrightarrow$$
$$4l^2+4l+1-10l-5=2k+1\Longrightarrow$$
$$4l^2+4l-10l-5=2k\Longrightarrow$$
$$4l^2+4l-10l-2k=5$$

If $n=2l+1,\ l$ is integer, then equation (4) does not have integer solutions either, because the left-hand side $(i.e. 4l^2+4l-10l-2k)$ is even, but the right-hand side $(i.e. 5)$ is odd.

It means that on the whole equation (4) does not have integer solutions, therefore, the term of (3) with z^{-1} will not be present, hence the coefficient d_{-1} of the term with $z^{-1}=\frac{1}{z}=\frac{1}{z^1}$ in (4) is zero, which leads to

$$Res_{z=0}(g)=Res_{z=0}(sinz\cdot f(z))=d_{-1}=0$$

(point $z_0=0$ is finite).

Coefficient d_{-1} can be computed as $d_{-1}=\frac{1}{2\pi i}\oint_T\frac{g(z)}{(z-0)^{-1+1}}=\frac{1}{2\pi i}\oint_T g(z)dz$, here T: |z| = (R/2).

Thus, by residue theorem, $\oint_T g(z)dz=2\pi i d_{-1}=2\pi i\cdot 0=0$ (we only deal with singularity 0).