Answer on Question #50230 – Math - Complex Analysis

Use Residue theorem to compute Principle value integral from (- infinity to infinity)
[3x] / [(x^2 + 1) ^2 (x+2)] dx

Solution

By definition, $P.V.\int_{-\infty}^{\infty}\frac{3zdz}{(z+2)(z^2+1)^2}=\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-2-\varepsilon}\frac{3zdz}{(z+2)(z^2+1)^2}+\int_{-2+\varepsilon}^{\infty}\frac{3zdz}{(z+2)(z^2+1)^2}\right)$

We use the contour consisting of the segment from $-R$ to $-2-\varepsilon$ along the real axis, followed by $-\gamma_\varepsilon$, the semicircle in the upper half plane, of radius ε. The contour $-\gamma_\varepsilon$ is clockwise, so that γ_ε is counter clockwise. Then we move from $-2+\varepsilon$ to R along the real axis, followed by γ_R, the semicircle of radius R in the upper half plane counter clockwise from R to $-R$. Call the whole contour $\Gamma_{\varepsilon,R}$. The contour $\Gamma_{\varepsilon,R}$ is closed in the upper half plane and encircles the pole of order 2 at $z=i$.

By the Residue Theorem,

$$\int_{\Gamma_{\varepsilon,R}}\frac{3zdz}{(z+2)(z^2+1)^2}=2\pi i\,\mathrm{res}[\frac{3z}{(z+2)(z^2+1)^2},z=i]=2\pi i\lim_{z\to i}\left(\frac{3z}{(z+2)(z+i)^2}\right)'=$$

$$=2\pi i\lim_{z\to i}\left(\frac{6(-z^2-z+i)}{(z+2)(z+i)^3}\right)=2\pi i(\frac{3}{25}+\frac{9}{100}i).$$

As for the top of the contour

$$\left|\int_{\Gamma_R}\frac{3zdz}{(z+2)(z^2+1)^2}\right|\le\frac{\pi R^2}{(R^2+1)^2(R+2)}\to 0 \text{ as } R\to\infty.$$

As for small semicircle, we parameterize γ_ε by $z(t)=-2+\varepsilon e^{it},\ 0\le t\le\pi.$

Then

$$\int_{\gamma_\varepsilon}\frac{3zdz}{(z+2)(z^2+1)^2}=3\int_0^\pi\frac{i\varepsilon e^{it}dt(-2+\varepsilon e^{it})}{\varepsilon e^{it}((-2+\varepsilon e^{it})^2+1)^2}=3i\int_0^\pi\frac{(-2+\varepsilon e^{it})}{(5-4\varepsilon e^{it}+\varepsilon^2e^{2it})^2}dt\to\frac{-6\pi i}{25}, \text{as } \varepsilon\to 0.$$

Combining all the parts, we have

$$2\pi i(\frac{3}{25}+\frac{9}{100}i)=\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{\Gamma_{R,\varepsilon}}\frac{3zdz}{(z+2)(z^2+1)^2}=P.V.\int_{-\infty}^{\infty}\frac{3zdz}{(z+2)(z^2+1)^2}-(-\frac{6\pi}{25}i).$$

Then, $P.V.\int_{-\infty}^{\infty}\frac{3zdz}{(z+2)(z^2+1)^2}=2\pi i(\frac{3}{25}+\frac{9}{100}i)-\frac{6\pi}{25}i=\frac{-9\pi}{50}.$

Answer: $\text{P.V.}\int_{-\infty}^{\infty}\frac{3z\,dz}{(z+2)(z^2+1)^2}=\frac{-9\pi}{50}.$