Answer on Question #50229 – Math – Complex Analysis

Use Residue theorem to compute
the integral from 0 to 2 pi
[Sin theta] / [4+ sin (theta) + cos (theta)] dtheta

Solution.

$$I=\int_0^{2\pi}\frac{\sin\theta}{4+\sin\theta+\cos\theta}d\theta=\left|\begin{array}{l} z=e^{i\theta},\ d\theta=\dfrac{dz}{iz} \\ \cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right) \\ \sin\theta=\dfrac{1}{2i}\left(z-\dfrac{1}{z}\right)\end{array}\right|=\oint_{|z|=1}\frac{\frac{1}{2i}\left(z-\frac{1}{z}\right)}{4+\frac{1}{2i}\left(z-\frac{1}{z}\right)+\frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz}=\oint_{|z|=1}f(z)dz,$$

where

$$f(z)=\frac{\frac{1}{2i}\left(z-\frac{1}{z}\right)}{\left(4+\frac{1}{2i}\left(z-\frac{1}{z}\right)+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)iz}=-\frac{z^2-1}{2z*z\left(4-\frac{i}{2}\left(z-\frac{1}{z}\right)+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}=-\frac{z^2-1}{z\left(8z-i\left(z^2-1\right)+\left(z^2+1\right)\right)}$$

$f(z)=\dfrac{z^2-1}{z\left[(i-1)z^2-8z-(1+i)\right]}$. Consider

$(i-1)z^2-8z-(1+i)=0$, hence $D=64+4(i-1)(i+1)=64+4\left(i^2-1\right)=64-8=56$, so

$$z_1=\frac{8+\sqrt{56}}{2(i-1)}=\frac{i+1}{(i-1)(i+1)}\left(4+\sqrt{14}\right)=\frac{i+1}{i^2-1}\left(4+\sqrt{14}\right)=\frac{i+1}{-2}\left(4+\sqrt{14}\right)=(1+i)\left(-2-\sqrt{\frac{7}{2}}\right)$$

$$z_2=\frac{8-\sqrt{56}}{2(i-1)}=\frac{i+1}{(i-1)(i+1)}\left(4-\sqrt{14}\right)=\frac{i+1}{i^2-1}\left(4-\sqrt{14}\right)=\frac{i+1}{-2}\left(4-\sqrt{14}\right)=(1+i)\left(-2+\sqrt{\frac{7}{2}}\right)$$

The isolated singular points of the function $f(z)$ are $z=\left\{0;(1+i)\left(-2\pm\sqrt{\frac{7}{2}}\right)\right\}$. All of them are simple poles. Only $z=0$ and $z=(1+i)\left(\sqrt{\frac{7}{2}}-2\right)$ are inside the contour $|z|=1$.

The residues in these points are the following:

$$\operatorname*{res}_{z=0} f=\lim_{z\to0}f\cdot z=\lim_{z\to0}\frac{z^2-1}{(i-1)z^2-8z-(1+i)}=\frac{-1}{-(1+i)}=\frac{1}{1+i}=\frac{1-i}{(1+i)(1-i)}=\frac{1-i}{1-i^2}=\frac{1-i}{1-(-1)}=\frac{1-i}{2}$$

,

$$\operatorname*{res}_{z=(1+i)\left(\sqrt{\frac{7}{2}}-2\right)} f=\lim_{z\to(1+i)\left(\sqrt{\frac{7}{2}}-2\right)} f\cdot\left[z-(1+i)\left(\sqrt{\frac{7}{2}}-2\right)\right]=\lim_{z\to(1+i)\left(\sqrt{\frac{7}{2}}-2\right)}\frac{z^2-1}{z(i-1)\left[z+(1+i)\left(\sqrt{\frac{7}{2}}+2\right)\right]}=$$

$$= \frac{(1+i)^2\left(\sqrt{\frac{7}{2}}-2\right)^2-1}{(i+1)(i-1)\left(\sqrt{\frac{7}{2}}-2\right)\left[(1+i)\left(\sqrt{\frac{7}{2}}-2\right)+(1+i)\left(\sqrt{\frac{7}{2}}+2\right)\right]} = \frac{2i\left(\sqrt{\frac{7}{2}}-2\right)^2-1}{-2\left(\sqrt{\frac{7}{2}}-2\right)\left[2(1+i)\left(\sqrt{\frac{7}{2}}-2\right)\right]} =$$

$$= \frac{2i\left(\sqrt{\frac{7}{2}}-2\right)^2-1}{-2\left(\sqrt{\frac{7}{2}}-2\right)\left[2(1+i)\left(\sqrt{\frac{7}{2}}-2\right)\right]} = (1-i)\frac{2i\left(\frac{7}{2}-2\cdot 2\sqrt{\frac{7}{2}}+4\right)-1}{-4\cdot 2\left(\frac{7}{2}-2\cdot 2\sqrt{\frac{7}{2}}+4\right)} = (1-i)\frac{2i\left(\frac{15}{2}-4\sqrt{\frac{7}{2}}\right)-1}{-4\cdot 2\left(\frac{15}{2}-4\sqrt{\frac{7}{2}}\right)} =$$

$$= (1-i)\frac{2i\left(\frac{15}{2}-4\sqrt{\frac{7}{2}}\right)-1}{-4\cdot 2\left(\frac{15}{2}-4\sqrt{\frac{7}{2}}\right)\left(\frac{15}{2}+4\sqrt{\frac{7}{2}}\right)}\left(\frac{15}{2}+4\sqrt{\frac{7}{2}}\right) = (1-i)\frac{2i\left(\frac{15}{2}-4\sqrt{\frac{7}{2}}\right)\left(\frac{15}{2}+4\sqrt{\frac{7}{2}}\right)-\left(\frac{15}{2}+4\sqrt{\frac{7}{2}}\right)}{-8\left(\frac{225}{4}-\frac{16\cdot 7}{2}\right)}$$

$$= (1-i)\frac{2i\left(\frac{225}{4}-\frac{16\cdot 7\cdot 2}{4}\right)-\left(\frac{15}{2}+4\sqrt{\frac{7}{2}}\right)}{-8\left(\frac{225}{4}-\frac{16\cdot 7\cdot 2}{4}\right)} = (1-i)\frac{\frac{i}{2}-\left(\frac{15}{2}+4\sqrt{\frac{7}{2}}\right)}{-2} = (1-i)\left(-\frac{i}{4}+\frac{15}{4}+2\sqrt{\frac{7}{2}}\right)$$

$$= (1-i)\left(-\frac{i}{4}+\frac{15}{4}+\frac{4\sqrt{14}}{4}\right) = -\frac{i}{4}+\frac{15}{4}+\frac{4\sqrt{14}}{4}-\frac{1}{4}-\frac{15}{4}i-\frac{4\sqrt{14}}{4}i = -\left(4+\sqrt{14}\right)i+\frac{7}{2}+\sqrt{14}.$$

According to the residue theorem,

$$I = 2\pi i\cdot\left[\underset{z=0}{res}\, f + \underset{z=(1+i)\left(\sqrt{\frac{7}{2}}-2\right)}{res} f\right] = 2\pi i\cdot\left(\frac{1-i}{2}-\left(4+\sqrt{14}\right)i+\frac{7}{2}+\sqrt{14}\right) = 2\pi i\cdot\left(-\left(\frac{9}{2}+\sqrt{14}\right)i+4+\sqrt{14}\right)$$

$$= \pi\cdot\left(\left(9+2\sqrt{14}\right)i+\left(8+2\sqrt{14}\right)i\right)$$

Answer: $\left(9+2\sqrt{14}\right)\pi+\left(8+2\sqrt{14}\right)\pi i$.