Answer on Question #50228 – Math – Complex Analysis

Use Cauchy Integral Formula to evaluate

Integral on Curve [e ^ (z+1)] / [{(z-i) (z^2 + (i-1)z-i) }^3]

Note : 1))please the figure is close curve I cannot paint it by writing but these points (3i,i,-i,-2i,-1) inside the figure if you need it when find the singularity inside the curve
2))Also I need all singularity (then sure only if inside curve i need all one with the working of cauchy integral , then plus it to find the total integral

Solution

$$\oint_C \frac{e^{z+1}}{(z-i)^3(z^2+(i-1)z-i)^3}dz$$

, where $C -$ closed curve.

Let us find singularities:

Numerator hasn't finite zeros. Denominator certainly has some, so let's find them.

$$(z-i)^3 = 0$$

$$z-i = 0$$

$$z_1 = i$$

$$(z^2+(i-1)z-i)^3 = 0$$

$$z^2+(i-1)z-i = 0$$

Due to the Vieta's formulas:

$$z_2 + z_3 = -\frac{i-1}{1} = 1-i$$

$$z_2 z_3 = -\frac{i}{1} = -i$$

It's obvious now that

$$z_2 = 1$$

$$z_3 = -i$$

As we said, numerator hasn't finite zeros, also $z_1 \neq z_2 \neq z_3$, thus, all these three zeros of denominator is nothing else, but poles of order 3.

$$\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = f^{(n)}(a)$$

– Cauchy's integral formula.

If area enclosed with a curve contains more than one singularity, we split up area/curve into parts, which contain each only one singularity.

$$\oint_C \frac{e^{z+1}}{(z-i)^3(z^2+(i-1)z-i)^3}dz =$$

$$\oint_C \frac{e^{z+1}}{(z-i)^3(z+i)^3(z-1)^3}dz =$$

$$\oint_{C_1} \frac{\frac{e^{z+1}}{(z+i)^3(z-1)^3}}{(z-i)^3}dz + \oint_{C_2} \frac{\frac{e^{z+1}}{(z-i)^3(z-1)^3}}{(z+i)^3}dz + \oint_{C_3} \frac{\frac{e^{z+1}}{(z-i)^3(z+i)^3}}{(z-1)^3}dz =$$

$$\frac{1}{\pi i}\oint_{C_1} \frac{\pi i\frac{e^{z+1}}{(z+i)^3(z-1)^3}}{(z-i)^{2+1}}dz + \frac{1}{\pi i}\oint_{C_2} \frac{\pi i\frac{e^{z+1}}{(z-i)^3(z-1)^3}}{(z+i)^{2+1}}dz + \frac{1}{\pi i}\oint_{C_3} \frac{\pi i\frac{e^{z+1}}{(z-i)^3(z+i)^3}}{(z-1)^{2+1}}dz =$$

$$\left(\pi i\frac{e^{z+1}}{(z+i)^3(z-1)^3}\right)''\Big|_{z=i} + \left(\pi i\frac{e^{z+1}}{(z-i)^3(z-1)^3}\right)''\Big|_{z=-i} +$$

$$\left(\pi i\frac{e^{z+1}}{(z-i)^3(z+i)^3}\right)''\Big|_{z=1} =$$

$$\pi i\frac{e^{z+1}\left(z^4-(14-2i)z^3+(60-22i)z^2-(40-68i)z-(7+24\,i)\right)}{(z-1)^5(z+i)^5}\Big|_{z=i} +$$

$$\pi i\frac{e^{z+1}\left(z^4-(14+2i)z^3+(60+22i)z^2-(40+68i)z-(7-24i)\right)}{(z-1)^5(z-i)^5}\Big|_{z=-i} +$$

$$\pi i\frac{e^{z+1}(z^4-12z^3+44z^2-12z-5)}{(z^2+1)^5}\Big|_{z=1} =$$

$$-\pi e^{1+i}\left(\frac{13}{32}+\frac{5}{8}i\right) + \pi e^{1-i}\left(\frac{13}{32}-\frac{5}{8}i\right) + \pi e^2\frac{i}{2}$$

It’s not clear if area contains $z=1$, so let think about $\pi e^2\frac{i}{2}$ like optional addendum.

Let us make some simplifications for the 1st and 2nd addenda:

$$-\pi e^{1+i}\left(\frac{13}{32}+\frac{5}{8}i\right) + \pi e^{1-i}\left(\frac{13}{32}-\frac{5}{8}i\right) =$$

$$\frac{13\pi e}{32}\left(-e^i+e^{-i}\right) - \frac{5i\pi e}{8}\left(e^i+e^{-i}\right) =$$

$$-\frac{13\pi e}{32}2i\sin(1) - \frac{5\pi e}{8}i2\cos(1) =$$

$$-\pi e\left(\frac{13}{16}\sin(1)+\frac{5}{4}\cos(1)\right)$$

Answer:

Singularities:

$$z_1 = i - pole\ of\ order\ 3$$

$$z_2 = 1 - pole\ of\ order\ 3$$

$$z_3 = -i - pole\ of\ order\ 3$$

z_1: $\frac{1}{\pi i}\oint_{C_1} \frac{\pi i \frac{e^{z+1}}{(z+i)^3(z-1)^3}}{(z-i)^{2+1}} dz = -\pi e^{1+i}\left(\frac{13}{32} + \frac{5}{8}i\right)$

z_2: $\frac{1}{\pi i}\oint_{C_3} \frac{\pi i \frac{e^{z+1}}{(z-i)^3(z+i)^3}}{(z-1)^{2+1}} dz = \pi e^2 \frac{i}{2}$

z_3: $\frac{1}{\pi i}\oint_{C_2} \frac{\pi i \frac{e^{z+1}}{(z-i)^3(z-1)^3}}{(z+i)^{2+1}} dz = \pi e^{1-i}\left(\frac{13}{32} - \frac{5}{8}i\right)$

Total integral:

$$\oint_C \frac{e^{z+1}}{(z-i)^3(z^2+(i-1)z-i)^3} dz = -\pi e\left(\frac{13}{16}\sin(1) + \frac{5}{4}\cos(1)\right) + \pi e^2 \frac{i}{2} (optional)$$