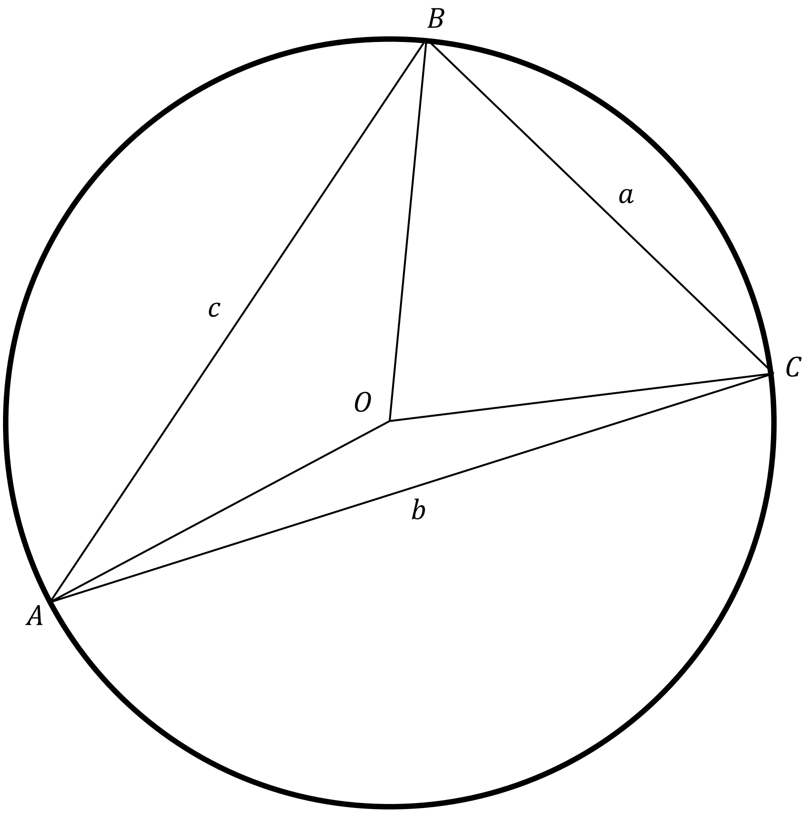
Answer on Question #50171 - Math - Trigonometry

In triangle ABC prove that $a^2 \cdot \cot A + b^2 \cdot \cot B + c^2 \cdot \cot C = 4 \cdot \Delta$, where Δ – area of the triangle.

Solution:



O is the center of the circumcircle. The area of the triangle AOB is given by the following formula

$$S_{AOB} = \frac{1}{2} AO \cdot OB \cdot \sin \widehat{AOB}$$

Since O is the center of the circumcircle, the angle $\widehat{AOB}$ two times larger than angle C of the triangle ABC. Using this and the facts that AO and OB are the radii of the circumcircle (we'll denote its value by R in the further) and that $\sin 2\alpha = 2 \sin \alpha \cdot \cos \alpha$, the previous formula can be rewritten in the following way

$$S_{AOB} = \frac{1}{2} AO \cdot OB \cdot \sin \widehat{AOB} = \frac{1}{2} R^2 \cdot \sin 2C = R^2 \cdot \sin C \cdot \cos C$$

The area of the triangle ABC can be given as a sum of areas of triangles AOB, BOC and COA:

$$\Delta = S_{AOB} + S_{BOC} + S_{COA} = R^2 \sin A \cdot \cos A + R^2 \sin B \cdot \cos B + R^2 \sin C \cdot \cos C$$

Now use the law of sines to express R in terms of sines of angles of triangle ABC and values of its sides.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$R^2 = \frac{1}{4}\left(\frac{a}{\sin A}\right)^2 = \frac{1}{4}\left(\frac{b}{\sin B}\right)^2 = \frac{1}{4}\left(\frac{c}{\sin C}\right)^2$$

Using this we obtain the following expression for Δ

$$\Delta = \frac{1}{4}\left(a^2\frac{\cos A}{\sin A} + b^2\frac{\cos B}{\sin B} + c^2\frac{\cos C}{\sin C}\right)$$

Multiplying this expression by 4 we obtain

$$a^2 \cdot \cot A + b^2 \cdot \cot B + c^2 \cdot \cot C = 4 \cdot \Delta,$$

as required.