Answer on Question #50127 - Math - Complex Analysis

Show that if $\sum z_n$ is convergent, then $\sum z_n^2$ is also convergent

Solution.

It cannot be shown, because it is false.

For example, let`s consider series $\sum z_n = \sum \frac{(-1)^n}{n^{1/4}}$. Since $\frac{1}{n^{1/4}}$ positive and decreasing to the 0, so by the Leibniz's theorem, the original series is convergent.

Now consider series $\sum z_n^2 = \sum \left(\frac{(-1)^n}{n^{1/4}} \right)^2 = \sum \frac{1}{n^{1/2}}$. By the Integral Test, the improper integral $\int_1^\infty \frac{dx}{x^{1/2}} = \lim_{A\to\infty} \int_1^A \frac{1}{x^{1/2}} dx = \lim_{A\to\infty} (2x^{1/2}) \Big|_1^A = \infty$ does not exist, therefore the series $\sum \frac{1}{n^{1/2}}$ is divergent.

This means that the original statement is not true.

Nevertheless, there exist some cases when the proposition is true.

For example, if we change our proposition like that: z_n are real, positive numbers and $\sum z_n$ is convergent, then $\sum z_n^2$ is also convergent.

Let's prove this variant of statement.

Since $\sum z_n$ is convergent, then by necessary condition, $\lim_{n\to\infty} |z_n| = 0$. That is why there exists a natural number N such that $\forall n > N \Rightarrow |z_n| < 1$. Then $\forall n > N \Rightarrow 0 \le |z_n^2| = |z_n|^2 \le |z_n|$, so from comparison test we obtain that $\sum z_n^2$ is convergent.