Answer on Question #50123 – Math - Complex Analysis

1) Show that:

$$\sum_{n=2}^{\infty}(-1)^n \cdot z^{2n+3} = \frac{z^7}{1+z^2}, \qquad |z|<1$$

Solution:

$$S = \sum_{n=2}^{\infty}(-1)^n \cdot z^{2n+3} = z^7 - z^9 + z^{11} - z^{13} + \ldots = \frac{z^7}{1+z^2}$$

It is a sum of descending geometrical progression

so $S = \dfrac{b_1}{1-q}$, where $b_1 = z^7$, $q = -z^2$

2) **a)** Is power series $\sum_{n=2}^{\infty}(-1)^n \cdot i^{2ni} \cdot i^{3i}$ convergent?

b) If yes, compute its sum

Solution:

a) $a_n = (-1)^n \cdot i^{2ni} \cdot i^{3i} = (-1)^n \cdot i^{(2n+3)i} = (-1)^n \cdot e^{Ln(i^{(2n+3)i})} = (-1)^n \cdot e^{(2n+3)i \cdot Ln(i)} =$

$$= (-1)^n \cdot e^{i\cdot(2n+3)\cdot[Ln|i| + i(Arg(i)+2\pi k)]} = \{k=0\} = (-1)^n \cdot e^{i\cdot(2n+3)\cdot[Ln 1 + i\cdot\frac{\pi}{2}]} = (-1)^n \cdot e^{-(2n+3)\cdot\frac{\pi}{2}}$$

$$|a_n| = e^{-(2n+3)\cdot\frac{\pi}{2}} = \frac{1}{e^{n\pi+\frac{3\pi}{2}}}$$

we can use the fact

$$\lim_{n\to\infty} a_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty}|a_n| = 0$$

$\lim_{n\to\infty}\dfrac{1}{e^{n\pi+\frac{3\pi}{2}}} = 0$, the necessary condition of convergence of series holds,

using Cauchy`s test (criterion)

$\lim_{n\to\infty}\sqrt[n]{\dfrac{1}{e^{n\pi+\frac{3\pi}{2}}}} = \lim_{n\to\infty}[\sqrt[n]{\dfrac{1}{e^{\frac{3\pi}{2}}}} \cdot \sqrt[n]{(\dfrac{1}{e^{\pi}})^n}] = \dfrac{1}{e^{\pi}} < 1 \quad \Rightarrow$ the series is convergent

Answer: convergent

b) Let`s compute the sum

$$S = \sum_{n=2}^{\infty} (-1)^n \cdot i^{2ni} \cdot i^{3i} = \sum_{n=2}^{\infty} (-1)^n \cdot e^{-(2n+3)\cdot\frac{\pi}{2}} = e^{-\frac{7\pi}{2}} - e^{-\frac{9\pi}{2}} + e^{-\frac{11\pi}{2}} - e^{-\frac{13\pi}{2}} + \ldots = \frac{e^{-\frac{7\pi}{2}}}{1+e^{-\pi}}$$

It is a sum of descending geometrical progression

so $S = \dfrac{b_1}{1-q}$ where $b_1 = e^{-\frac{7\pi}{2}}$, $q = -e^{-\pi}$

Answer: $S = \dfrac{e^{-\frac{7\pi}{2}}}{1+e^{-\pi}}$