Answer on Question #50122 – Math – Complex Analysis

f (z) = [(e^z) - 1] / [z^3 (1+{z^2})]

A- Compute the residue of f at z= o by using laurent series

B- Use the previous to classify the singularity z=0

C- Without Computing the laurent series ,Classify the singularity z =i

Solution.

A. $e^z = 1 + z + \frac{z^2}{2} + O(z^3)$, $\frac{1}{1+z^2} = 1 - z^2 + z^4 + O(z^6)$

$$f(z) = \frac{e^z - 1}{z^3(1+z^2)} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2} + O(z^3) - 1\right)\left(1 - z^2 + z^4 + O(z^6)\right) =$$

$$= \left(\frac{1}{z^2} + \frac{1}{2z} - 1 - \frac{z}{2} + O(z^2)\right) = \frac{c_{-n}}{(z-a)^n} + \cdots + \frac{c_{-1}}{z-a} + \sum_{k=0}^{\infty} c_k (z-a)^k$$

Thus, $Res(f, 0) = \frac{1}{2}$, **because the residue** $Res\, f(a)$ **of function** $f(z)$ **at a singularity** a **is equal to the coefficient of** $\frac{1}{z-a}$ **in the Laurent expansion of** $f(z)$ **in a neighbourhood of the point** a **(here** $a = 0$**, coefficient is equal to 1/2).**

B. $\lim_{z \to 0} z^2 f(z) = \lim_{z \to 0} z^2 \left(\frac{1}{z^2} + \frac{1}{z} - 1 - \frac{z}{2} + O(z^2)\right) = 1$.

Thus, the singularity $z = 0$ **is the pole of order 2.**

Other method is the following: the principal part of the Laurent expansion of $f(z)$ **around** $a = 0$ **contains finitely many terms if and only if a point** $a = 0$ **is a pole of function** $f(z)$**. If** $c_{-n} \neq 0$ $(c_{-n} = c_{-2} = 1)$ **in Laurent expansion, then the order of the pole** $a = 0$ **of the function** $f(z)$ **is equal to** $n = 2$**.**

Thus, the singularity $z = 0$ **is the pole of order 2.**

C. $\lim_{z\to i} f(z) = \frac{e^i-1}{i^3}\lim_{z\to i}\frac{1}{1+z^2} = \frac{e^i-1}{i^3}\lim_{z\to i}\frac{1}{(1+iz)(1-iz)} =$
$\frac{e^i-1}{2i^3}\lim_{z\to i}\frac{1}{-i^2+iz} = \frac{e^i-1}{2i^3}\lim_{z\to i}\frac{i}{i-z} = \infty.$

It means that the singularity $z = i$ is the pole.

Besides, $\lim_{z\to i}(z-i)\, f(z) = \frac{e^i-1}{2}$

Thus, the singularity $z = i$ is the pole of order 1.

Other method is the following:

the function $f(z) = \frac{e^z-1}{z^3(1+z^2)} = \frac{e^z-1}{z^3(z-i)(z+i)} = \frac{\frac{e^z-1}{z^3(z+i)}}{z-i} = \frac{h(z)}{z-i}$**, where**

$h(z) = \frac{e^z-1}{z^3(z+i)}$, $\quad h(i) = \frac{e^i-1}{i^3(i+i)} = \frac{cos(1)+isin1-1}{2i^4} = \frac{cos(1)-1}{2} + i\frac{sin(1)}{2} \neq 0$, **is such that the singularity $z = i$ is a simple zero of function**

$\frac{1}{f(z)} = \frac{z^3(1+z^2)}{e^z-1} = \frac{z^3(z-i)(z+i)}{e^z-1}$**, hence, the singularity $z = i$ is the pole of order 1 of function** $f(z) = \frac{e^z-1}{z^3(1+z^2)}$