Answer to Question #49513 – Math – Differential Calculus | Equations

So, we have a next problem:

find the solution of an ordinary differential equation

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x, \qquad (1)$$

that satisfies initial conditions (a Cauchy problem):

$$y(0) = 1,\ y'(0) = 2 \qquad (2)$$

Solution

The first step: Let's find the particular solution y_h of homogeneous equation :

$$y'' - 3y' + 2y = 0$$

Let's write the characteristic equation:

$$\lambda^2 - 3\lambda + 2 = 0.$$

Solving this equation we obtain two values of λ, namely $\lambda_1 = 1$ and $\lambda_2 = 2$.

So, $y_h = C_1e^x + C_2e^{2x}$, where C_1 and C_2 are unknown coefficients, which will be determined below.

The second step: Taking into account the right-hand side of equation (1), let's find the particular solution y_p of the equation (1) :

$$y_p = A\cos(3x) + B\sin(3x) + Ce^{-x},$$

where A, B and C are unknown coefficients. Let's find these coefficients.

$$y_p' = -3A\sin(3x) + 3B\cos(3x) - Ce^{-x}$$

$$y_p'' = -9A\cos(3x) - 9B\sin(3x) + Ce^{-x}.$$

Substituting y_p, y_p', y_p'' for y and its derivatives in the initial equation (1), we will obtain the next three equations to determine A, B and C:

$$-9A\cos(3x) - 9B\sin(3x) + Ce^{-x} - 3(-3A\sin(3x) + 3B\cos(3x) - Ce^{-x}) + 2(A\cos(3x) + B\sin(3x) + Ce^{-x}) =$$
$$= 3e^{-x} - 10\cos 3x$$

Now let's collect the coefficients for $\cos(3x)$:

$$(-9A - 9B + 2A) = -10.$$

Similarly, collect the coefficients for $\sin(3x)$:

$$-9B + 9A + 2B = 0$$

In the same way, for e^{-x}:

$C+3C+2C=3 \Rightarrow\ C=\frac{1}{2}.$

Solving the system of equations:

$$\begin{cases} -7A-9B=-10 \\ -7B+9A=0 \end{cases},$$

we will find $A=\frac{7}{13}, B=\frac{9}{13}.$

Thus, $y_p=\frac{7}{13}\cos(3x)+\frac{9}{13}\sin(3x)+\frac{1}{2}e^{-x}$.

The third step*:* Let's write the general solution of (1) as y_h+y_p:

$$y=y_h+y_p=C_1e^x+C_2e^{2x}+\frac{7}{13}\cos(3x)+\frac{9}{13}\sin(3x)+\frac{1}{2}e^{-x}. \qquad (3)$$

The fourth step: Recall conditions (2) and solve the Cauchy problem, determine the coefficients C_1 and C_2. From (3) we obtain

$$y'=C_1e^x+2C_2e^{2x}-\frac{21}{13}\sin(3x)+\frac{27}{13}\cos(3x)-\frac{1}{2}e^{-x} \qquad (4)$$

Plug (3) and (4) into (2):

$$\begin{cases} y(0)=1 \\ y'(0)=2 \end{cases} \Rightarrow \begin{cases} C_1e^0+C_2e^0+\frac{7}{13}+\frac{1}{2}e^0=1 \\ C_1e^0+2C_2e^0+\frac{27}{13}-\frac{1}{2}e^0=2 \end{cases} \Rightarrow \begin{cases} C_1+C_2+\frac{7}{13}+\frac{1}{2}=1 \\ C_1+2C_2+\frac{27}{13}-\frac{1}{2}=2 \end{cases} \Rightarrow \begin{cases} C_1=-\frac{1}{2} \\ C_2=\frac{6}{13} \end{cases}$$

So, $y=-\frac{1}{2}e^x+\frac{6}{13}e^{2x}+\frac{7}{13}\cos(3x)+\frac{9}{13}\sin(3x)+\frac{1}{2}e^{-x}$.

Answer: $y=-\frac{1}{2}e^x+\frac{6}{13}e^{2x}+\frac{7}{13}\cos(3x)+\frac{9}{13}\sin(3x)+\frac{1}{2}e^{-x}$.