Answer on Question #49511 - Math – Differential Calculus | Equations

Solve the initial value problem by using Laplace transformation.

$$y'' - 6y' + 15y = 2\sin 3t \qquad y(0) = -1, y'(0) = -4$$

<u>Solution:</u>

It is known that the Laplace transformation of the n^{th} derivative of the function $f(t)$ is given by

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0)$$

where $F(s)$ is the Laplace transformation of $f(t)$.

So the Laplace transformation of $y'' - 6y' + 15y$ has the following form

$$\mathcal{L}\{y'' - 6y' + 15y\}(s) = s^2 Y(s) - s \cdot y(0) - y'(0) - 6\big(sY(s) - y(0)\big) + 15Y(s)$$

where $Y(s)$ is the Laplace transformation of $y(t)$. Substituting $y(0) = -1$ and $y'(0) = -4$ into this equation we obtain

$$\mathcal{L}\{y'' - 6y' + 15y\}(s) = (s^2 - 6s + 15)Y(s) + s - 2$$

The Laplace transformation of the function $\sin(a \cdot t)$ is given by

$$\mathcal{L}\{\sin(a \cdot t)\}(s) = \frac{a}{s^2 + a^2}$$

Thus the Laplace transformation of $2\sin 3t$ is given by

$$\mathcal{L}\{2\sin 3t\}(s) = \frac{6}{s^2 + 9}$$

Therefore the Laplace transformation of the initial equation gives us the following

$$(s^2 - 6s + 15)Y(s) + s - 2 = \frac{6}{s^2 + 9}$$

Expressing $Y(s)$ from this equation we obtain

$$Y(s) = \frac{6}{(s^2 + 9)(s^2 - 6s + 15)} - \frac{s - 2}{s^2 - 6s + 15} \qquad (1)$$

Before using the inverse Laplace transformation we decompose the first term in the right-hand side of this equation into partial fractions. It is easy to verify that this decomposition has the following form

$$\frac{6}{(s^2+9)(s^2-6s+15)} = \frac{1}{10}\frac{s+1}{s^2+9} - \frac{1}{10}\frac{s-5}{s^2-6s+15}$$

Substituting this into equation (1) we obtain

$$Y(s) = \frac{1}{10}\frac{s+1}{s^2+9} - \frac{11s-25}{s^2-6s+15}$$

or, which is equivalent to

$$Y(s) = \frac{1}{10}\frac{s}{s^2+9} + \frac{1}{30}\frac{3}{s^2+9} - \frac{11}{10}\frac{s-3}{(s-3)^2+6} - \frac{4}{5\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}$$

Considering the fact that

$$\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}(t) = \sin \omega t$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}(t) = \cos \omega t$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+\alpha)^2+\omega^2}\right\}(t) = e^{-\alpha t}\sin \omega t$$

$$\mathcal{L}^{-1}\left\{\frac{s+\alpha}{(s+\alpha)^2+\omega^2}\right\}(t) = e^{-\alpha t}\cos \omega t$$

we can easily find the inverse Laplace transformation of $Y(s)$

$$y = \frac{1}{10}\cos 3t + \frac{1}{30}\sin 3t - \left(\frac{11}{10}\cos\sqrt{6}t + \frac{4}{5\sqrt{6}}\sin\sqrt{6}t\right)e^{3t}$$

Answer: $\frac{1}{10}\cos 3t + \frac{1}{30}\sin 3t - \left(\frac{11}{10}\cos\sqrt{6}t + \frac{4}{5\sqrt{6}}\sin\sqrt{6}t\right)e^{3t}$.