**Answer on Question #49494 – Math– Differential Calculus | Equations**

**Question:**

Solve the second order differential equation

**y'' - 3y' + 2y = 3e^(-x) - 10cos3x y(0) = 1 y'(0) =2**

**Solution:**

Solve $-3\frac{dy(x)}{dx}+\frac{d^2y(x)}{dx^2}+2\,y(x)=3\,e^{-x}-10\cos(3\,x)$, such that $y(0)=2$ and $y'(0)=2$:

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The general solution will be the sum of the complementary solution and particular solution.

Find the complementary solution by solving $\frac{d^2y(x)}{dx^2}-3\frac{dy(x)}{dx}+2\,y(x)=0$:

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Assume a solution will be proportional to $e^{\lambda x}$ for some constant $\lambda$.

Substitute $y(x)=e^{\lambda x}$ into the differential equation:

$$\frac{d^2}{dx^2}\left(e^{\lambda x}\right)-3\frac{d}{dx}\left(e^{\lambda x}\right)+2\,e^{\lambda x}=0$$

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Substitute $\frac{d^2}{dx^2}\left(e^{\lambda x}\right)=\lambda^2 e^{\lambda x}$ and $\frac{d}{dx}\left(e^{\lambda x}\right)=\lambda\,e^{\lambda x}$:

$$\lambda^2 e^{\lambda x}-3\,\lambda\,e^{\lambda x}+2\,e^{\lambda x}=0$$

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Factor out $e^{\lambda x}$:

$$\left(\lambda^2-3\,\lambda+2\right)e^{\lambda x}=0$$

Since $e^{\lambda x} \neq 0$ for any finite $\lambda$, the zeros must come from the polynomial:
$\lambda^2 - 3\lambda + 2 = 0$

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Factor:
$(\lambda - 2)(\lambda - 1) = 0$

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Solve for $\lambda$:
$\lambda = 1$ or $\lambda = 2$

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The root $\lambda = 1$ gives $y_1(x) = c_1 e^x$ as a solution, where $c_1$ is an arbitrary constant.
The root $\lambda = 2$ gives $y_2(x) = c_2 e^{2x}$ as a solution, where $c_2$ is an arbitrary constant.
The general solution is the sum of the above solutions:
$y(x) = y_1(x) + y_2(x) = c_1 e^x + c_2 e^{2x}$

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Determine the particular solution to $\frac{d^2 y(x)}{dx^2} + 2\,y(x) - 3\,\frac{dy(x)}{dx} = 3\,e^{-x} - 10\cos(3x)$ by the method of undetermined coefficients:
The particular solution will be the sum of the particular solutions to
$\frac{d^2 y(x)}{dx^2} + 2\,y(x) - 3\,\frac{dy(x)}{dx} = 3\,e^{-x}$ and $\frac{d^2 y(x)}{dx^2} + 2\,y(x) - 3\,\frac{dy(x)}{dx} = -10\cos(3x)$.

The particular solution to $\frac{d^2 y(x)}{dx^2} + 2\,y(x) - 3\,\frac{dy(x)}{dx} = 3\,e^{-x}$ is of the form:

$y_{p_1}(x) = \frac{a_1}{e^x}$

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The particular solution to $\frac{d^2 y(x)}{dx^2} + 2\,y(x) - 3\,\frac{dy(x)}{dx} = -10\cos(3x)$ is of the form:
$y_{p_2}(x) = a_2\cos(3x) + a_3\sin(3x)$

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Sum $y_{p_1}(x)$ and $y_{p_2}(x)$ to obtain $y_p(x)$:
$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \frac{a_1}{e^x} + a_2\cos(3x) + a_3\sin(3x)$

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Solve for the unknown constants $a_1$, $a_2$, and $a_3$:
Compute $\frac{dy_p(x)}{dx}$:

$$\begin{aligned}\frac{dy_p(x)}{dx} &= \frac{d}{dx}\left(\frac{a_1}{e^x} + a_2\cos(3x) + a_3\sin(3x)\right)\\ &= -\frac{a_1}{e^x} - 3\,a_2\sin(3x) + 3\,a_3\cos(3x)\end{aligned}$$

Compute $\frac{d^2 y_p(x)}{dx^2}$:

$$\begin{aligned}\frac{d^2 y_p(x)}{dx^2} &= \frac{d^2}{dx^2}\left(\frac{a_1}{e^x} + a_2 \cos(3x) + a_3 \sin(3x)\right)\\ &= \frac{a_1}{e^x} - 9 a_2 \cos(3x) - 9 a_3 \sin(3x)\end{aligned}$$

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Substitute the particular solution $y_p(x)$ into the differential equation:

$$\frac{d^2 y_p(x)}{dx^2} - 3\frac{d y_p(x)}{dx} + 2 y_p(x) = \frac{3}{e^x} - 10\cos(3x)$$

$$\left(\frac{a_1}{e^x} - 9 a_2 \cos(3x) - 9 a_3 \sin(3x)\right) - 3\left(-\frac{a_1}{e^x} - 3 a_2 \sin(3x) + 3 a_3 \cos(3x)\right) + 2\left(\frac{a_1}{e^x} + a_2 \cos(3x) + a_3 \sin(3x)\right) = \frac{3}{e^x} - 10\cos(3x)$$

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Simplify:

$$\frac{6 a_1}{e^x} + (-7 a_2 - 9 a_3)\cos(3x) + (9 a_2 - 7 a_3)\sin(3x) = \frac{3}{e^x} - 10\cos(3x)$$

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Equate the coefficients of $e^{-x}$ on both sides of the equation:

$6 a_1 = 3$

Equate the coefficients of $\cos(3x)$ on both sides of the equation:

$-7 a_2 - 9 a_3 = -10$

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Equate the coefficients of $\sin(3x)$ on both sides of the equation:

$9 a_2 - 7 a_3 = 0$

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Solve the system:

$a_1 = \frac{1}{2}$

$a_2 = \frac{7}{13}$

$a_3 = \frac{9}{13}$

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Substitute $a_1$, $a_2$, and $a_3$ into $y_p(x) = a_1 e^{-x} + a_3 \sin(3x) + a_2 \cos(3x)$:

$$y_p(x) = \frac{1}{2 e^x} + \frac{7}{13}\cos(3x) + \frac{9}{13}\sin(3x)$$

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The general solution is:

$$y(x) = y_c(x) + y_p(x) = \frac{1}{2 e^x} + \frac{7}{13}\cos(3x) + \frac{9}{13}\sin(3x) + c_1 e^x + c_2 e^{2x}$$

Solve for the unknown constants using the initial conditions:

Compute $\frac{dy(x)}{dx}$:

$$\frac{dy(x)}{dx} = \frac{d}{dx}\left(\frac{1}{2e^x} + \frac{7}{13}\cos(3x) + \frac{9}{13}\sin(3x) + c_1 e^x + c_2 e^{2x}\right)$$
$$= -\frac{1}{2e^x} + \frac{27}{13}\cos(3x) - \frac{21}{13}\sin(3x) + c_1 e^x + 2c_2 e^{2x}$$

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Substitute $y(0) = 2$ into $y(x) = c_1 e^x + c_2 e^{2x} + \frac{e^{-x}}{2} + \frac{9}{13}\sin(3x) + \frac{7}{13}\cos(3x)$:

$$c_1 + c_2 + \frac{27}{26} = 2$$

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Substitute $y'(0) = 2$ into $\frac{dy(x)}{dx} = c_1 e^x + 2c_2 e^{2x} - \frac{e^{-x}}{2} - \frac{21}{13}\sin(3x) + \frac{27}{13}\cos(3x)$:

$$c_1 + 2c_2 + \frac{41}{26} = 2$$

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Solve the system:

$c_1 = \frac{3}{2}$

$c_2 = -\frac{7}{13}$

Substitute $y(0) = 2$ into $y(x) = c_1\, e^x + c_2\, e^{2x} + \frac{e^{-x}}{2} + \frac{9}{13}\sin(3x) + \frac{7}{13}\cos(3x)$:

$$c_1 + c_2 + \frac{27}{26} = 2$$

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Substitute $y'(0) = 2$ into $\frac{dy(x)}{dx} = c_1\, e^x + 2\, c_2\, e^{2x} - \frac{e^{-x}}{2} - \frac{21}{13}\sin(3x) + \frac{27}{13}\cos(3x)$:

$$c_1 + 2\, c_2 + \frac{41}{26} = 2$$

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Solve the system:

$c_1 = \frac{3}{2}$

$c_2 = -\frac{7}{13}$

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Substitute $c_1 = \frac{3}{2}$ and $c_2 = -\frac{7}{13}$ into $y(x) =$
$c_1\, e^x + c_2\, e^{2x} + \frac{e^{-x}}{2} + \frac{9}{13}\sin(3x) + \frac{7}{13}\cos(3x)$:

Answer:

$$y(x) = -\frac{7}{13}\, e^{2x} + \frac{3}{2}\, e^x + \frac{e^{-x}}{2} + \frac{9}{13}\sin(3x) + \frac{7}{13}\cos(3x)$$