Answer on Question #49481 – Math – Complex Analysis

Test series for convergence:

1) [i^n/(2^(n+2))]

2) (n!)^2/e^n

3) 1/[root(i+n)]^n

4) e^(i coshn)

5) [n+i/(4^n)]

Solution

The complex series $\sum_{n=0}^{\infty} c_n$ is said to converge absolutely if the real series $\sum_{n=0}^{\infty} |c_n|$ converges.
The following statement can be proved : if a complex series converges absolutely, then it converges.
We shall use this fact in the next problems.

1) $\left|\frac{i^n}{2^{n+2}}\right| = \frac{1}{2^{n+2}} = \frac{1}{4} \cdot \frac{1}{2^n}$ is a geometric sequence with common ratio $q = \frac{a_{n+1}}{a_n} = \frac{1}{2} < 1$, so the series converges.

2) $\frac{\boldsymbol{c_{n+1}}}{\boldsymbol{c_n}} = \frac{\left((\boldsymbol{n+1})!\right)^2}{\boldsymbol{e^{n+1}}} : \frac{(\boldsymbol{n}!)^2}{\boldsymbol{e^n}} = \frac{(\boldsymbol{n+1})^2}{\boldsymbol{e}} > 1$ for all $n \geq 1$, $\lim_{n\to\infty} \frac{\boldsymbol{c_{n+1}}}{\boldsymbol{c_n}} = \lim_{n\to\infty} \frac{(n+1)^2}{e} > 1$ (it is plus infinity), hence by d' Alambert`s ratio test, the series diverges..

3) $\left|\frac{1}{\left(\sqrt{i+n}\right)^n}\right| = \left|\frac{1}{(i+n)^{\frac{n}{2}}}\right| = \frac{1}{|i+n|^{\frac{n}{2}}} < \frac{1}{n^{\frac{n}{2}}}$. Note that $\sqrt[n]{\frac{1}{n^{\frac{n}{2}}}} = \frac{1}{\sqrt{n}} < 1$, $\lim_{n\to\infty} \sqrt{|a_n|} < \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^{\frac{n}{2}}}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and by the Cauchy ratio test, the series converges.

4) $\left|\boldsymbol{e^{icosh(n)}}\right| = 1 \nrightarrow 0$ (it does not tend to zero) as $n \to \infty$ (the necessary condition of convergence does not hold true in this case), so the series diverges.

5) $\left|\frac{n+i}{4^n}\right| < \frac{2n}{4^n}$. Note that $\lim_{n\to\infty} \sqrt[n]{\frac{2n}{4^n}} = \frac{1}{4} < 1$ and by the Cauchy ratio test, the series converges.