Answer on Question #49468 – Math – Analytic Geometry

A circle has the equation $x^2 + y^2 + 12x + 8y + 7 = 0$

a) Write down the coordinates of the center C.

b) Show that the line with equation $x = 13 - 2y$ is a tangent to the circle.

c) Find the coordinates of the point Q where the tangent and circle meet.

d) The point P lies on the given line such that PQ=QC. Find the two possible coordinates of P.

Solution:

a) The general form of the equation for a circle with its center at (h, k) and radius of r is:

$$(x - h)^2 + (y - k)^2 = r^2$$

where the center being at the point (h, k) and the radius being "r".

$$x^2 + y^2 + 12x + 8y + 7 = 0$$

Then we take the constant to the right side.

$$x^2 + y^2 + 12x + 8y = -7$$

Now we can group the y-terms and x-terms.

$$(x^2 + 12x) + (y^2 + 8y) = -7$$

Then we convert the left side to squared form.

$$(x^2 + 12x + 36) - 36 + (y^2 + 8y + 16) - 16 = -7$$

Simplify the obtained expression.

$$(x + 6)^2 + (y + 4)^2 = 36 + 16 - 7$$

Simplify the right side of the equation.

$$(x + 6)^2 + (y + 4)^2 = 45$$

Compare the last equation to general form of the equation for a circle $(x - h)^2 + (y - k)^2 = r^2$. We see that h = -6, k = -4 and r = $\sqrt{45} \approx 6.708$ units. So the center of the circle is (-6,-4) and the radius is 6.708 units. As a result of our calculation we can represent the graph of the found equation for circle. The graph is shown on Figure 1.

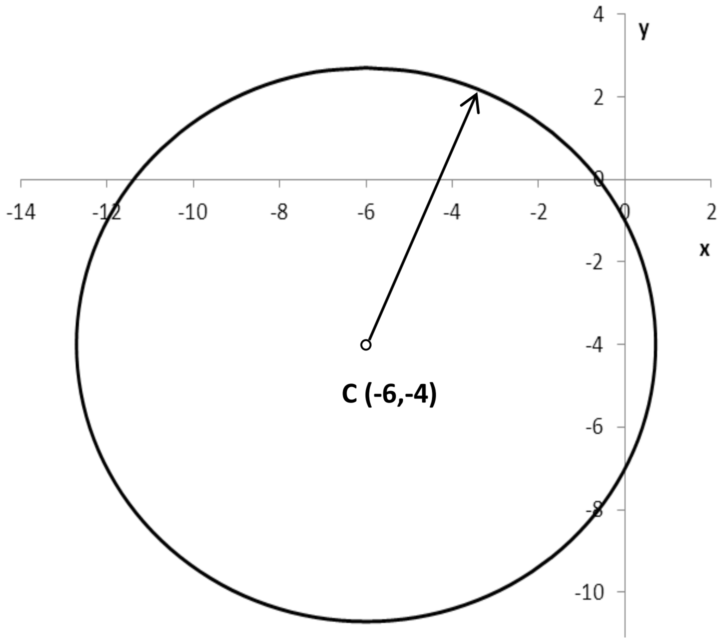

Figure 1 The graph of the equation $x^2 + y^2 + 12x + 8y + 7 = 0$.

b) Show that the line with equation $x = 13 - 2y$ is a tangent to the circle.

We apply the method by substitution. Thus we have the following.

$$(13 - 2y)^2 + y^2 + 12(13 - 2y) + 8y + 7 = 0$$

Simplify the obtained equation by opening the parenthesis.

$$169 - 52y + 4y^2 + y^2 + 156 - 24y + 8y + 7 = 0$$

Combine like terms.

$$5y^2 - 68y + 332 = 0$$

We obtained the quadratic equation, so we can find the discriminant.

$$D = b^2 - 4ac$$

We have the following coefficients: a = 5, b = -68 and c = 332. Substitute into the formula.

$$D = (-68)^2 - 4(5)(332) = 4624 - 6640 = -2016$$

Then we can determine the root of the equation.

$$y_1 = \frac{68 + \sqrt{-2016}}{10} = \frac{34}{5} + \frac{6}{5}\sqrt{14i}$$

$$y_2 = \frac{68 - \sqrt{-2016}}{10} = \frac{34}{5} - \frac{6}{5}\sqrt{14i}$$

We can also consider another method of solution by showing that line to be a tangent to the circle is when the length of perpendicular from the center of the circle equals the radius of the circle.

Firstly we find the equation of the slope. We rewrite the original equation.

$$13 - 2y = x$$

Add -13 to both sides of the equation.

$$-2y = x - 13$$

Divide both sides by -2.

$$y = -\frac{1}{2}x + 6.5$$

From the slope-intercept form of the line, I can see that the slope of the given line is $-\frac{1}{2}$, so radius line has slope m = 2. We plug this and the center point into the point-slope equation of a straight line.

$$y - (-4) = 2(x - (-6))$$

We open the parenthesis.

$$y + 4 = 2x + 12$$

$$y = 2x + 8$$

c) The intersection of the radius line and the tangent line is a point on the circle, we solve the system of equations represented by these two lines:

$$-\frac{1}{2}x + 6.5 = 2x + 8$$

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Combine like terms on the left and right sides of the equation.

$$-2.5x = 1.5$$

Divide both sides by -2.5 and find the value of x.

$$x = -0.6 = -\frac{3}{5}$$

Now we determine the value of y.

$$y = 2(-0.6) + 8 = 6.8$$

The point where the tangent and circle meet (-0.6, 6.8). But this point will relate to the circle with another equation.

The point Q where the tangent and circle meet has the following coordinates (-4.835, 2.606).

Now we can determine the line of the point Q (-4.835, 2.606) which lies on the circle.

Firstly we find the slope = $\frac{2.606+4}{-4.835+6} = \frac{6.606}{1.165} = 5.670$

Hence $m_{tgt} = -0.1764$

Now we can determine the equation of the tangent at Q, which is equal.

$$y - 2.606 = -0.1764(x - (-4.835))$$

$$y = -0.1764x - 0.8527 + 2.606$$

$$y = -0.1764x + 1.753$$

The graph is shown on Figure 2.

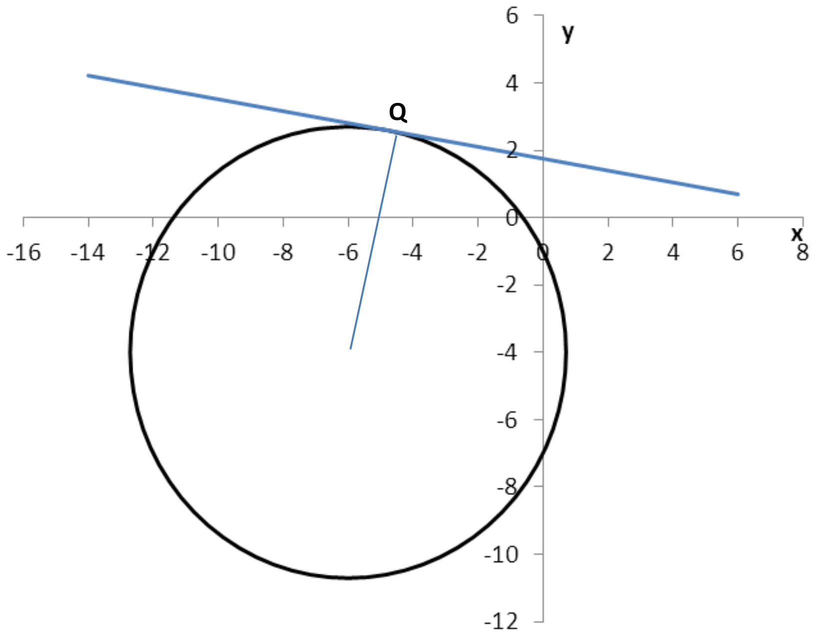

Figure 2 The graph of the equation $(x+6)^2 + (y+4)^2 = 45$ and the line $y = -0.1764x + 1.753$.

d) The point P lies on the given line such that PQ=QC. Find the two possible coordinates of P.

We have already determined the coordinate of the point Q (-4.835, 2.606). Based on the noted information that PQ=QC we can conclude the Q is midpoint of the line PC. To find the coordinate of the point P we apply the following formula.

$$Q\ (-4.835, 2.606) = (\frac{X_P + X_C}{2}, \frac{Y_P + Y_C}{2})$$

$$\frac{X_P + X_C}{2} = -4.835$$

$$\frac{Y_P + Y_C}{2} = 2.606$$

Simplify the expressions.

$$X_P + X_C = -9.67$$

$$Y_P + Y_C = 5.212$$

$$X_P = -9.67 - X_C$$

$$Y_P = 5.212 - Y_C$$

Finally the coordinates of the point will be equal to $(-9.67 - X_C, 5.212 - Y_C)$.