Answer on Question #49462 – Math – Multivariable Calculus

$$\begin{cases} x^2 - y^2 + 2uv + 15 = 0 \\ x + 2xy - u^2 + v^2 - 10 = 0 \end{cases}$$

1st step. Let us derive both equations with respect to x.

$$\frac{\partial}{\partial x}(x^2 - y^2 + 2uv + 15) = \frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}\frac{\partial y}{\partial x} + 2\left(v\frac{\partial u}{\partial u}\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial v}\frac{\partial v}{\partial x}\right) + \frac{\partial 15}{\partial y} =$$

$$2x - 2y\frac{\partial y}{\partial x} + 2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x} + 0 = 0$$

$$x - y\frac{\partial y}{\partial x} + v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} = 0 \quad (1)$$

$$\frac{\partial}{\partial x}(x + 2xy - u^2 + v^2 - 10) = \frac{\partial x}{\partial x} + 2\left(y\frac{\partial x}{\partial x} + x\frac{\partial y}{\partial x}\right) - \frac{\partial u^2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial v^2}{\partial v}\frac{\partial v}{\partial x} - \frac{\partial 10}{\partial x} =$$

$$1 + 2y + 2x\frac{\partial y}{\partial x} - 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} - 0 = 0$$

$$\frac{1}{2} + y + x\frac{\partial y}{\partial x} - u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad (2)$$

2nd step. Let us derive both equations with respect to y.

$$\frac{\partial}{\partial y}(x^2 - y^2 + 2uv + 15) = \frac{\partial x^2}{\partial x}\frac{\partial x}{\partial y} - \frac{\partial y^2}{\partial y} + 2\left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) + \frac{\partial 15}{\partial y} =$$

$$2x\frac{\partial x}{\partial y} - 2y + 2u\frac{\partial v}{\partial y} + 2v\frac{\partial u}{\partial y} + 0 = 0$$

$$-y + x\frac{\partial x}{\partial y} + v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} = 0 \quad (3)$$

$$\frac{\partial}{\partial y}(x + 2xy - u^2 + v^2 - 10) = \frac{\partial x}{\partial y} + 2\left(x\frac{\partial y}{\partial y} + y\frac{\partial x}{\partial y}\right) - \frac{\partial u^2}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial v^2}{\partial v}\frac{\partial v}{\partial y} - \frac{\partial 10}{\partial y} =$$

$$\frac{\partial x}{\partial y} + 2x + 2y\frac{\partial x}{\partial y} - 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} - 0 = 0$$

$$x + \left(\frac{1}{2} + y\right)\frac{\partial x}{\partial y} - u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad (4)$$

3rd step. Factor out $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

$$\frac{\partial u}{\partial x} = \frac{-x + y\frac{\partial y}{\partial x} - u\frac{\partial v}{\partial x}}{v} \quad (5)$$

$$\frac{\partial u}{\partial x} = \frac{\frac{1}{2} + y + x\frac{\partial y}{\partial x} + v\frac{\partial v}{\partial x}}{u} \quad (6)$$

$$\frac{\partial u}{\partial y} = \frac{y - x\frac{\partial x}{\partial y} - u\frac{\partial v}{\partial y}}{v} \quad (7)$$

$$\frac{\partial u}{\partial y} = \frac{x + \left(\frac{1}{2} + y\right)\frac{\partial x}{\partial y} + v\frac{\partial v}{\partial y}}{u} \quad (8)$$

4th step. Use info from 3rd step.

$$\frac{-x + y\frac{\partial y}{\partial x} - u\frac{\partial v}{\partial x}}{v} = \frac{\frac{1}{2} + y + x\frac{\partial y}{\partial x} + v\frac{\partial v}{\partial x}}{u}$$

$$-xu + yu\frac{\partial y}{\partial x} - u^2\frac{\partial v}{\partial x} = \frac{1}{2}v + yv + xv\frac{\partial y}{\partial x} + v^2\frac{\partial v}{\partial x}$$

$$\left(-xu - \frac{1}{2}v - yv\right) + (yu - xv)\frac{\partial y}{\partial x} = (u^2 + v^2)\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{\left(-xu - \frac{1}{2}v - yv\right) + (yu - xv)\frac{\partial y}{\partial x}}{u^2 + v^2} \quad (9)$$

$$\frac{y - x\frac{\partial x}{\partial y} - u\frac{\partial v}{\partial y}}{v} = \frac{x + \left(\frac{1}{2} + y\right)\frac{\partial x}{\partial y} + v\frac{\partial v}{\partial y}}{u}$$

$$yu - xu\frac{\partial x}{\partial y} - u^2\frac{\partial v}{\partial y} = xv + \left(\frac{1}{2} + y\right)v\frac{\partial x}{\partial y} + v^2\frac{\partial v}{\partial y}$$

$$(yu - xv) - \left(xu + \frac{1}{2}v + yv\right)\frac{\partial x}{\partial y} = (u^2 + v^2)\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{(yu - xv) - \left(xu + \frac{1}{2}v + yv\right)\frac{\partial x}{\partial y}}{u^2 + v^2} \quad (10)$$

5th step. Substitute result of 4th step into 3rd.

$$\frac{\partial u}{\partial x} = \frac{-x + y\frac{\partial y}{\partial x} - u\left(\frac{\left(-xu - \frac{1}{2}v - yv\right) + (yu - xv)\frac{\partial y}{\partial x}}{u^2 + v^2}\right)}{v} =$$

$$\frac{(-xu^2 - xv^2) + (yu^2 + yv^2)\frac{\partial y}{\partial x} + \left(xu^2 + \frac{1}{2}uv + yuv\right) + (-yu^2 + xuv)\frac{\partial y}{\partial x}}{v(u^2 + v^2)} =$$

$$\frac{\left(-xv^2 + \frac{1}{2}uv + yuv\right) + (yv^2 + xuv)\frac{\partial y}{\partial x}}{v(u^2 + v^2)} =$$

$$\frac{\left(-xv + \frac{1}{2}u + yu\right) + (yv + xu)\frac{\partial y}{\partial x}}{(u^2 + v^2)}$$

$$\frac{\partial u}{\partial x} = \frac{\left(\frac{1}{2}u + yu - xv\right) + (yv + xu)\frac{\partial y}{\partial x}}{u^2 + v^2} \quad (11)$$

$$\frac{\partial u}{\partial y} = \frac{y - x\frac{\partial x}{\partial y} - u\left(\frac{(yu - xv) - \left(xu + \frac{1}{2}v + yv\right)\frac{\partial x}{\partial y}}{u^2 + v^2}\right)}{v} =$$

$$\frac{(yu^2 + yv^2) + (-xu^2 - xv^2)\frac{\partial x}{\partial y} + (-yu^2 + xuv) + \left(xu^2 + \frac{1}{2}uv + yuv\right)\frac{\partial x}{\partial y}}{v(u^2 + v^2)} =$$

$$\frac{(yv^2 + xuv) + \left(\frac{1}{2}uv + yuv - xv^2\right)\frac{\partial x}{\partial y}}{v(u^2 + v^2)} =$$

$$\frac{(yv + xu) + \left(\frac{1}{2}u + yu - xv\right)\frac{\partial x}{\partial y}}{u^2 + v^2}$$

$$\frac{\partial u}{\partial y} = \frac{(yv + xu) + \left(\frac{1}{2}u + yu - xv\right)\frac{\partial x}{\partial y}}{u^2 + v^2} \quad (12)$$

6th step. Let us make some transformation with (11).

$$\frac{\partial u}{\partial x} = \frac{u(1 + 2y) - 2xv + 2(yv + xu)\frac{\partial y}{\partial x}}{2(u^2 + v^2)} \quad (11')$$

7th step. Compare (11′) and (12) with desirable answer.

$$\frac{\partial u}{\partial x} = \frac{u(1 + 2y) - 2xv}{2(u^2 + v^2)} \quad (13)$$

$$\frac{\partial u}{\partial y} = \frac{yv + xu}{u^2 + v^2} \quad (14)$$

As soon as we suppose $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ not to be equal to zero, the only way is to set $\frac{\partial y}{\partial x}$ and $\frac{\partial x}{\partial y}$ equal to zero (i.e. we suppose that x and y are independent one from another).

Note: simplification $\frac{\partial y}{\partial x} = \frac{\partial x}{\partial y} = 0$ is not necessary.