Answer on Question #49459 – Math – Multivariable Calculus

Evaluate

2 2x

a) ∫ ∫ (4x+2)dydx

0 x^2

1 y √1-y^2

b) ∫ ∫ ∫ z dzdxdy

0 0 0

1 √1-x^2

c) ∫ ∫ (x^2+y^2)^3/2 dydx

-1 0

Solution

a)

$$\int_0^2 dx \int_{x^2}^{2x} dy(4x+2) = \int_0^2 (4x(2x\text{-}x^2) + 4x\text{-}2x^2)dx = \int_0^2 (8x^2\text{-}4x^3 + 4x\text{-}2x^2)dx =$$

$$= \int_0^2 (6x^2\text{-}4x^3 + 4x)dx = (\text{-}x^4 + 2x^3 + 2x^2)\Big|_0^2 = 8$$

b)

$$\int_0^1 dy \int_0^y dx \int_0^{\sqrt{1-y^2}} z\,dz = \int_0^1 dy \int_0^y dx \frac{z^2}{2}\Big|_0^{\sqrt{1-y^2}} = \frac{1}{2}\int_0^1 dy \int_0^y dx\,(1-y^2) = \frac{1}{2}\int_0^1 dy\,(1-y^2)\,x\Big|_0^y = \frac{1}{2}\int_0^1 dy\ (1-y^2)\,y = \frac{1}{2}\int_0^1 dy\,(y-y^3) =$$

$$= \frac{1}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{8}$$

c)

$$I = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy\,(x^2+y^2)^{3/2}.$$

Let's look at a set of inequalities for x and y that describe this region.

$$-1 \le x \le 1$$
$$0 \le y \le \sqrt{1-x^2}.$$

The region is obviously a half of a circle or semicircle of radius 1 (see Fig.1).

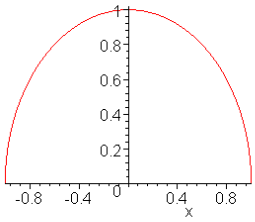

Fig.1.

Due to the limits on the inner integral this is liable to be time-consuming integral to compute.

However, a half of the semicircle of radius 1 can be defined in polar coordinates by the following inequalities:

$0 \le \rho \le 1$ $0 \le \varphi \le \pi$,	where $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, ρ is the radius-vector; $x^2 + y^2 = \rho^2$, φ is the polar angle, $\tan(\varphi) = \frac{y}{x}$, $0 \le \varphi \le 2\pi$.

$|I| = \rho$ is Jacobian, $dxdy = \rho d\rho d\varphi$,

So, it is very simply to compute this integral in polar coordinates:

$(x^2 + y^2)^{3/2} = \rho^3$.

$$\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} dy\ (x^2+y^2)^{3/2} = \left| \begin{matrix} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ 0 \le \rho \le 1 \\ 0 \le \varphi \le \pi \\ I = \rho \end{matrix} \right| = \int_0^{\pi} d\varphi \int_0^1 \rho\, d\rho\, \rho^3 = \int_0^{\pi} d\varphi \int_0^1 \rho^4 d\rho = \int_0^{\pi} d\varphi \left. \frac{\rho^5}{5} \right|_0^1 = \frac{1}{5}\int_0^{\pi} d\varphi = \frac{1}{5}\pi.$$