Answer on Question #49456 – Math– Multivariable Calculus

Question:

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)dxdy \tag{1}$$

and write an equivalent integral with the order of integration reversed. Hence, evaluate the integral.

Solution:

1) Suppose that the the region of integration *D* is defined by $x_1 \le x \le x_2$ and $y_1(x) \le y \le y_2(x)$ (the vertically simple region). The double integral is given by

$$\iint_{(D)} f(x,y)dxdy = \int_{x_1}^{x_2} \left(\int_{y_1(x)}^{y_3(x)} f(x,y)dy \right) dx \tag{2}$$

and is computed as iterated integral.

2) If the region *D* is defined by $y_1 \le y \le y_2$ and $x_1(y) \le x \le x_2(y)$ (the horizontally simple region), then the double integral is given by

$$\iint_{(D)} f(x,y)dxdy = \int_{y_1}^{y_2} \left(\int_{x_1(y)}^{x(y)} f(x,y)dx \right) dy \quad , \tag{3}$$

and also is evaluated as iterated integral.

In our case the region *D* is bounded by the curves $y(x)=x^2$ and $y(x)=2x$ (see fig.1)

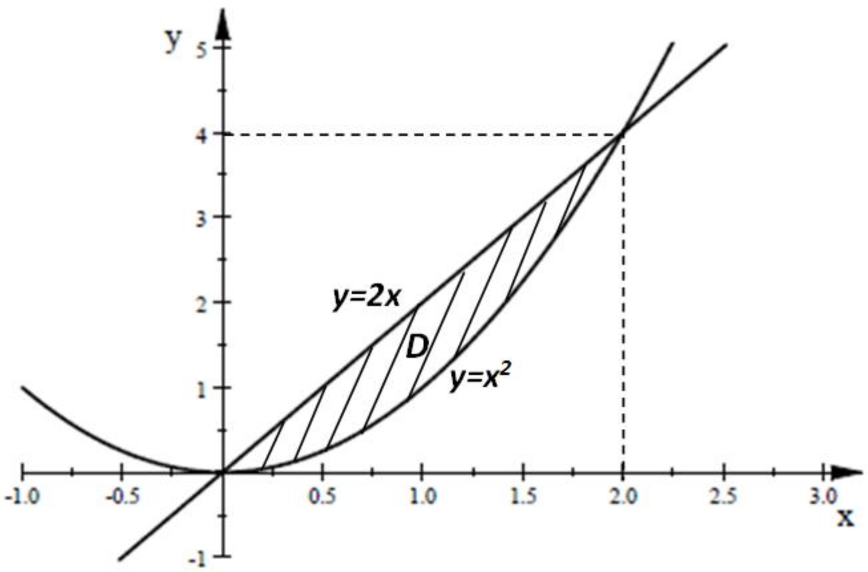

Fig.1

In the original integral, the integration order is *dydx (*this order corresponds to integrating first with respect to y*)*. So, to compute (1) we can use the form (2) assuming that $0 \le x \le 2$ and $x^2 \le y \le 2x$:

$$\int_0^2 \left(\int_{x^2}^{2x} (4x+2)dy \right) dx = \int_0^2 \left((4xy+2y)|_{x^2}^{2x} \right) dx = \int_0^2 \left((4x \cdot 2x + 2 \cdot 2x - (4x \cdot x^2 + 2 \cdot x^2)) \right) dx$$

$$= \int_0^2 (8x^2 + 4x - (4x^3 + 2x^2))\, dx = \int_0^2 (6x^2 + 4x - 4x^3)\, dx = \left(6\frac{x^3}{3} + 4\frac{x^2}{2} - 4\frac{x^4}{4} \right) |_0^2$$

$$= (2^4 + 2^3 - 2^4) = 8$$

Now let us change the order of integration in (1) to be *dxdy.* For this we transform the limits of integration into the domain *D.* The order *dxdy* means that *y* is the variable of the outer integral and its limits must be constant. So for variable *y* (see fig.1) we have 0$\le y \le 4$. As *x* will be the variable for the inner integration, we need to integrate with respect to *x* first. We can rewrite the equations *y(x)=x*^{2} and *y(x)=2x as* $x(y) = \sqrt{y}$ and $x(y) = \frac{y}{2}$ respectively. Hence, the range of *x* is $\frac{y}{2} \le x \le \sqrt{y}$. Therefore, the integral (1) with the reversed order of integration takes the form (3), where region *D* is described by pair of inequalities 0$\le y \le 4$ and $\sqrt{y} \le x \le \frac{y}{2}$. Let us evaluate it.

$$\int_0^4 \left(\int_{\frac{y}{2}}^{\sqrt{y}} (4x+2)dx \right) dy = \int_0^4 \left(\left(4\frac{x^2}{2} + 2x \right) |_{\frac{y}{2}}^{\sqrt{y}} \right) dy = \int_0^4 \left(2\left(\sqrt{y}\right)^2 + 2 \cdot \sqrt{y} - 2\left(\frac{y}{2}\right)^2 - 2 \cdot \frac{y}{2} - \right) dy$$

$$= \int_0^4 \left(y + 2 \cdot \sqrt{y} - \frac{y^2}{2} \right) dy = \left(\frac{y^2}{2} + 2 \cdot \frac{2}{3}\sqrt{y^3} - \frac{y^3}{6} \right) |_0^4 = \left(\frac{4^2}{2} + 2 \cdot \frac{2}{3}\sqrt{4^3} - \frac{4^3}{6} \right)$$

$$= \left(8 + \frac{2^6}{6} - \frac{2^6}{6} \right) = 8.$$

As it must, these iterated integrals give the same answer.

Answer: $\int_0^2 \left(\int_{x^2}^{2x} (4x+2)dy \right) dx = \int_0^4 \left(\int_{\frac{y}{2}}^{\sqrt{y}} (4x+2)dx \right) dy = 8.$