Answer on Question # 49124 – Math - Calculus

Determine whether the series convergent or divergent and find its sum if exist

$$\sum_{n=1}^{\infty}\left(\frac{5}{3^{\sqrt{n}}}+\frac{1}{n^2}\right).$$

Solution.

Consider the series

$$\sum_{n=1}^{\infty}\left(\frac{5}{3^{\sqrt{n}}}+\frac{1}{n^2}\right).$$

Two functions $y = 5n^2$ and $y = 3^{\sqrt{n}}$ intersect at two points (approximately under $n \approx 0.71$ and $n \approx 95.26$).

Fix $N = 96$ and for all $n \geq N$ we have $3^{\sqrt{n}} \geq 5n^2$, hence $\frac{5}{3^{\sqrt{n}}} \leq \frac{1}{n^2}$.

We show that

$$\frac{5}{3^{\sqrt{n}}}<\frac{1}{n^2}$$

as $n\to\infty$:

$$\lim_{x\to\infty}\frac{\frac{5}{3^{\sqrt{x}}}}{\frac{1}{x^2}}=5\lim_{x\to\infty}\frac{x^2}{3^{\sqrt{x}}}=\left[\frac{\infty}{\infty}\right]=5\lim_{x\to\infty}\frac{\left(x^2\right)'}{\left(3^{\sqrt{x}}\right)'}=5\lim_{x\to\infty}\frac{2x}{3^{\sqrt{x}}\ln 3\frac{1}{2\sqrt{x}}}=5\lim_{x\to\infty}\frac{4x^{3/2}}{3^{\sqrt{x}}\ln 3}=$$

$$=20\lim_{x\to\infty}\frac{\left(x^{3/2}\right)'}{\left(3^{\sqrt{x}}\ln 3\right)'}=20\lim_{x\to\infty}\frac{3/2x^{1/2}}{3^{\sqrt{x}}\ln^2 3\frac{1}{2\sqrt{x}}}=20\lim_{x\to\infty}\frac{3x}{3^{\sqrt{x}}\ln^2 3}=60\lim_{x\to\infty}\frac{x'}{\left(3^{\sqrt{x}}\ln^2 3\right)'}=$$

$$=60\lim_{x\to\infty}\frac{1}{3^{\sqrt{x}}\ln^3 3\frac{1}{2\sqrt{x}}}=120\lim_{x\to\infty}\frac{\sqrt{x}}{3^{\sqrt{x}}\ln^3 3}=120\lim_{x\to\infty}\frac{\left(\sqrt{x}\right)'}{\left(3^{\sqrt{x}}\ln^3 3\right)'}=120\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{3^{\sqrt{x}}\ln^4 3\frac{1}{2\sqrt{x}}}=$$

$$=120\lim_{x\to\infty}\frac{1}{3^{\sqrt{x}}\ln^4 3}=0<1.$$

So

$$\sum_{n=1}^{\infty}\left(\frac{5}{3^{\sqrt{n}}}+\frac{1}{n^2}\right)<2\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Next we use the integral test.

Let

$$f(x)=\frac{1}{x^2}.$$

Consider the integral

$$\int_1^{\infty}\frac{1}{x^2}dx=\lim_{M\to\infty}\int_1^{M}x^{-2}dx=\lim_{M\to\infty}\left.\frac{x^{-1}}{-1}\right|_1^M=-\lim_{M\to\infty}\left.\frac{1}{x}\right|_1^M=$$

$$=-\lim_{M\to\infty}\left(\frac{1}{M}-1\right)=1<\infty.$$

Thus, the integral

$$\int_1^{\infty}\frac{1}{x^2}dx$$

converges.

Hence by the Integral Test, the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges.

Therefore series

$$\sum_{n=1}^{\infty}\left(\frac{5}{3^{\sqrt{n}}}+\frac{1}{n^2}\right)$$

converges too.

Answer: The series $\sum_{n=1}^{\infty}\left(\frac{5}{3^{\sqrt{n}}}+\frac{1}{n^2}\right)$ converges.