Answer on Question #49119 – Math – Calculus

Determine whether the series convergent or divergent $\sum_{n=1}^{\infty} \frac{5^n}{n!}$

Solution.

Denote $a_n = \frac{5^n}{n!}$. Hence,

$$a_{n+1} = a_n \cdot \frac{5}{n+1};$$

$$n \geq 5 \Rightarrow \frac{5}{n+1} \leq \frac{5}{6};$$

So

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{4} a_n + a_5 + a_6 + a_7 + \cdots \leq \sum_{n=1}^{4} a_n + a_5 + \frac{5}{6} a_5 + \left(\frac{5}{6}\right)^2 a_5 + \cdots =$$

$$= \sum_{n=1}^{4} a_n + a_5 \left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots \right) = \sum_{n=1}^{4} a_n + \frac{a_5}{1 - \frac{5}{6}} = 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!\left(1 - \frac{5}{6}\right)}$$

$$= 220.625 < \infty.$$

We use the first comparison criterion. If $0 \leq a_n \leq b_n$ (starting from some), then convergence of the series $\sum_{n=1}^{\infty} b_n$ implies convergence of $\sum_{n=1}^{\infty} a_n$. Here $a_n = \frac{5^n}{n!}$, $b_n = \frac{5^5}{5!}\left(\frac{5}{6}\right)^n$.

Thus, the series $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ is convergent.