Answer on Querstion #49111 - Math - Complex Analysis:

Find an integral $\int_0^{2\pi} e^{e^{i\theta}} d\theta$.

Solution

We can't use all of the standard tricks and tools of integration like substitution, trig substitution, parts.I thinks the best choice is to use Taylor formula. Let's denote $e^{\theta} = t$ and we will have the next e^t.

Now we will use Taylor series and we will have the next expression:

$$1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

Now let's integrate our expression and we will have the next:

$$\int_0^{2\pi} e^{e^{i\theta}} d\theta = \int_0^{2\pi} \left(1 + \frac{e^{i\theta}}{1!} + \frac{e^{2i\theta}}{2!} + \frac{e^{3i\theta}}{3!} + \cdots\right) d\theta = 1\Big|_0^{2\pi} = 2\pi$$

Here we will count integral with power $n = 2k + 1$ but we will show it for $k = 0$

$$\int_0^{2\pi} \frac{e^{i\theta}}{1!} d\theta = \frac{1}{1!}\left(\int_0^{2\pi} \cos\theta d\theta + i\int_0^{2\pi} \sin\theta d\theta\right) = \frac{1}{1!}\left(\sin\theta\Big|_0^{2\pi} - \cos\theta\Big|_0^{2\pi}\right) =$$

$$= \frac{1}{1!}(\sin 2\pi - \sin 0 - i(\cos 2\pi - \cos 0)) = 0$$

Here we will count integral with power $n = 2k$ but we will show it for $k = 1$

$$\int_0^{2\pi} \frac{e^{2i\theta}}{2!} d\theta = \frac{1}{2!}\left(\int_0^{2\pi} \cos 2\theta d\theta + i\int_0^{2\pi} \sin 2\theta d\theta\right) = \frac{1}{2!}\frac{1}{2}\left(\sin 2\theta\Big|_0^{2\pi} - \cos 2\theta\Big|_0^{2\pi}\right) =$$

$$= \frac{1}{2!}\frac{1}{2}(\sin 4\pi - \sin 0 - i(\cos 4\pi - \cos 0)) = 0$$

So we don't have any expressions with factorials.

And our integral will be next $\int_0^{2\pi} e^{e^{i\theta}} d\theta = 2\pi$