Answer on Question #48983 – Math – Multivariable Calculus

Use Lagrange multipliers to find the max and min values of the function

f(x,y) = (x^2)(y^2)(z^2)

subject to the constraint function x^2 + y^2 + z^2 = 1.

Solution.

We find extrema of the function $f = x^2 y^2 z^2$ and the constraint is $g = x^2 + y^2 + z^2 - 1 = 0$,

$\vec{\nabla} f = \left(2xy^2z^2; 2x^2yz^2; 2x^2y^2z\right)$, $\vec{\nabla} g = (2x; 2y; 2z)$.

Lagrange`s function is

$L = x^2y^2z^2 + \lambda(x^2 + y^2 + z^2 - 1)$

Calculate

$L_x = 0$,

$L_y = 0$,

$L_z = 0$,

$L_\lambda = 0$,

i.e.

$2xy^2z^2 + 2\lambda x = 0$,

$2x^2yz^2 + 2\lambda y = 0$,

$2x^2y^2z + 2\lambda z = 0$,

$x^2 + y^2 + z^2 - 1 = 0$,

Then,

$2x(y^2z^2 + \lambda) = 0$,

$2y(x^2z^2 + \lambda) = 0$,

$2z(x^2y^2 + \lambda) = 0$,

$x^2 + y^2 + z^2 = 1$,

Observe that if any one of *x*, *y*, *z* is zero, then

$$f = 0.$$

If none of the variables is equal to zero, then the equations become

$$\begin{cases} y^2z^2 = -\lambda \\ x^2z^2 = -\lambda \\ x^2y^2 = -\lambda \end{cases}$$

and because *x*; *y*; $z \neq 0$, this indicates that $y^2z^2 = x^2z^2 = x^2y^2$, so $|x| = |y| = |z|$. Then the constraint becomes $3x^2 = 1$, or $x = \pm\frac{1}{\sqrt{3}}$. In this case,

$$f\left(\pm\frac{1}{\sqrt{3}}; \pm\frac{1}{\sqrt{3}}; \pm\frac{1}{\sqrt{3}}\right) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27}.$$

So, the maximum $f = \frac{1}{27}$ occurs at $\left(\pm\frac{1}{\sqrt{3}}; \pm\frac{1}{\sqrt{3}}; \pm\frac{1}{\sqrt{3}}\right)$, where all combinations of signs "+" and "–" are possible (there will be 8 cases). The minimum $f = 0$ occurs at any point (*x*; *y*; *z*), obeying the constraint $x^2 + y^2 + z^2 = 1$, where one of the variables is zero.

Answer: $f_{\max} = \frac{1}{27}$, $f_{\min} = 0$.