Answer on Question #48980 – Math –Multivariable Calculus

Find the local maximum and minimum values and saddle point or points of the function

$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

Solution:

To find and classify critical points we find all of the first and second derivatives of order

$$\frac{\partial f}{\partial x} = 6x^2 + y^2 + 10x$$

$$\frac{\partial f}{\partial y} = 2xy + 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 12x + 10$$

$$\frac{\partial^2 f}{\partial y^2} = 2x + 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y$$

Critical points will be solutions to the system of equations:

$$\frac{\partial f}{\partial x} = 6x^2 + y^2 + 10x = 0$$

$$\frac{\partial f}{\partial y} = 2xy + 2y = 0$$

It follows $y = 0$ or $x = -1$ from the second equation.

If $y = 0$, then from the first equation we get

$2x(3x + 5) = 0$, hence $x = 0$ or $x = -\frac{5}{3}$.

If $x = -1$, then from the first equation we get

$y^2 - 4 = 0$, hence $y = 2$ or $y = -2$.

The critical points are $(0,0)$, $\left(-\frac{5}{3}, 0\right)$ $(-1,-2)$, $(-1,2)$. At these points:

$(0,0)$

$$\frac{\partial^2 f}{\partial x^2} = 10$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$D(0,0) = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 20 > 0$$

So, $D(0,0)$ is positive and $\frac{\partial^2 f}{\partial x^2}$ is positive and so the point $(0,0,0)$is a point of local minimum

$$\left(-\frac{5}{3},0\right)$$

$$\frac{\partial^2 f}{\partial x^2} = -10$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{4}{3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$D\left(-\frac{5}{3},0\right) = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \frac{40}{3} > 0$$

So, $D\left(-\frac{5}{3},0\right)$ is positive and $\frac{\partial^2 f}{\partial x^2}$ is negative and so the point $\left(-\frac{5}{3},0,\frac{125}{27}\right)$is a point of local maximum

$$(-1,-2)$$

$$\frac{\partial^2 f}{\partial x^2} = 12x + 10 = -2$$

$$\frac{\partial^2 f}{\partial y^2} = 2x + 2 = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2y = -4$$

$$D(-1,-2) = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -16 < 0$$

So, $D(-1,-2)$ is negative and so the point $(-1,-2,3)$is a saddle point

$$(-1,2)$$

$$\frac{\partial^2 f}{\partial x^2} = 12x + 10 = -2$$

$$\frac{\partial^2 f}{\partial y^2} = 2x + 2 = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2y = 4$$

$$D(-1,2) = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -16 < 0$$

So, $\boldsymbol{D(-1,2)}$ is negative and so the point $(-1,2,3)$is a saddle point