Determine whether the series convergent or divergent by using integral test

$$\sum_{n=2}^{\infty} \frac{2\ln(n-1)}{n-1}$$

Solution.

We use the integral test

Let $f(x) = \dfrac{2\ln(x-1)}{x-1}$. Consider the integral $\displaystyle\int_{2}^{\infty} \frac{2\ln(x-1)}{x-1} dx$.

$$\int_{2}^{\infty} \frac{2\ln(x-1)}{x-1} dx = \begin{bmatrix} \ln(x-1) = t \\ dt = \dfrac{dx}{x-1} \\ x=2 \Rightarrow t=0 \\ x=\infty \Rightarrow t=\infty \end{bmatrix} = \int_{0}^{\infty} 2t\,dt = \lim_{M\to\infty} \int_{0}^{M} 2t\,dt = \lim_{M\to\infty} \frac{2t^2}{2}\Big|_{0}^{M} = \lim_{M\to\infty} t^2\Big|_{0}^{M} = \lim_{M\to\infty}\left(M^2 - 0\right) = \infty.$$

Thus the integral $\displaystyle\int_{2}^{\infty} \frac{2\ln(x-1)}{x-1} dx$ diverges.

Hence by the Integral Test the series $\displaystyle\sum_{n=2}^{\infty} \frac{2\ln(n-1)}{n-1}$ diverges.

Answer: The series $\displaystyle\sum_{n=2}^{\infty} \frac{2\ln(n-1)}{n-1}$ diverges.