Answer on Question #48854 – Math – Calculus

If $0<\theta<\frac{\pi}{2}$, then find the minimum value of $\tan\theta+\cot\theta$.

Solution

Method 1

$$f(\theta)=\tan\theta+\cot\theta$$

$$f'(\theta)=\frac{1}{\cos^2\theta}+\left(-\frac{1}{\sin^2\theta}\right)=\frac{\sin^2\theta-\cos^2\theta}{\sin^2\theta\cdot\cos^2\theta}=-\frac{\cos 2\theta}{\sin^2\theta\cdot\cos^2\theta}$$

$$f'(\theta)=0\ \Rightarrow -\frac{\cos 2\theta}{\sin^2\theta\cdot\cos^2\theta}=0$$

$$\cos 2\theta=0$$

$$2\theta=\frac{\pi}{2}+\pi n, n\in\mathbb{Z}$$

$$\theta=\frac{\pi}{4}+\frac{\pi n}{2}, n\in\mathbb{Z}$$

Take into account condition $0<\theta<\frac{\pi}{2}$ and obtain

$$\theta=\frac{\pi}{4}$$

On the borders (0 and $\frac{\pi}{2}$) $f(\theta)=\infty$.

The minimum value:

$$f\left(\frac{\pi}{4}\right)=\tan\frac{\pi}{4}+\cot\frac{\pi}{4}=1+1=2$$

Method 2

If $0<\theta<\frac{\pi}{2}$, then $\tan\theta>0, \cot\theta>0$. Let $\tan\theta=a, a>0$.

Use inequality of arithmetic and geometric means:

$$\tan\theta+\cot\theta=\tan\theta+\frac{1}{\tan\theta}\geq 2\sqrt{\tan\theta\cdot\frac{1}{\tan\theta}}=2,$$

equality sign is possible if and only if $\tan\theta = \frac{1}{\tan\theta}$, i.e. $\tan\theta = 1$

(we choose only positive value of $\tan\theta$), hence

$$\theta = \frac{\pi}{4}$$

The minimum value is $\tan\frac{\pi}{4} + \cot\frac{\pi}{4} = 1 + 1 = 2$

Answer: the minimum value is 2.