Answer on Question #48727 – Math – Statistics and Probability

If X and Y are i.i.d r.v.s with $f_X\ (x) = \alpha \cdot e^{-\alpha x}\ u(x)$ and $f_Y\ (y) = \alpha \cdot e^{-\alpha y} u(y)$. Find the pdf $f_Z\ (z)$ for

$Z = 2X + Y$.

Solution

$F_{2X}(t) = P(2X < t) = P\left(X < \frac{t}{2}\right) = F_X\left(\frac{t}{2}\right)$,

Find probability distribution function of $2X$:

$f_{2X}(t) = \frac{d}{dt} F_{2X}(t) = \frac{d}{dt} F_X\left(\frac{t}{2}\right) = \frac{1}{2} f_X\left(\frac{t}{2}\right) = \frac{\alpha}{2} \cdot e^{-\frac{\alpha t}{2}}\ u\left(\frac{t}{2}\right)$, where $u(t) = 1$ when $t \geq 0$;

$u(t) = 0$ when $t < 0$.

The probability density function of the sum of two independent random variables $2X$ and Y, each of which has a probability density function, is the convolution of their separate density functions:

$$f_Z\ (z) = f_{2X+Y}(z) = \int_{-\infty}^{+\infty} f_{2X}(x) f_Y(z-x) dx = \int_{-\infty}^{+\infty} \frac{\alpha}{2} \cdot e^{-\frac{\alpha x}{2}}\ u\left(\frac{x}{2}\right) \alpha \cdot e^{-\alpha(z-x)} u(z-x)) dx =$$

$$= \frac{\alpha^2}{2} \int_0^z e^{-\alpha z + \frac{\alpha x}{2}} dx = \alpha e^{-\alpha z}\ e^{\frac{\alpha x}{2}}\Big|_0^z u(z) = \alpha e^{-\alpha z} \left(e^{\frac{\alpha z}{2}} - 1\right) u(z) = \alpha e^{-\frac{\alpha z}{2}} \left(1 - e^{-\frac{\alpha z}{2}}\right) u(z).$$