Answer on Question #48725-Math-Statistics and Probability

Given the following Y=X·H, where X and H are two random variables with pdf

$f_X(x) = \frac{1}{x} e^{-x^2}$,x≥0 ,and $f_H(h) = \frac{1}{h} e^{-3h^2}$

What is the pdf $f_{Y|X}(y|x)$?

What is the pdf $f_Y(y)$?

Solution

$$f_{XH}(x,h) = f_X(x) f_H(h) = \frac{1}{x} e^{-x^2} \cdot \frac{1}{h} e^{-3h^2} = f_{X\frac{Y}{X}}\left(x, \frac{y}{x}\right).$$

$$f_{XY}(x,y) = f_{X\frac{Y}{X}}\left(x, \frac{y}{x}\right) \cdot |J|,$$

where

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial\left(\frac{y}{x}\right)}{\partial x} & \frac{\partial\left(\frac{y}{x}\right)}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{x}.$$

So

$$f_{XY}(x,y) = \frac{1}{x} \cdot \frac{1}{x} e^{-x^2} \cdot \frac{1}{\left(\frac{y}{x}\right)} e^{-3\left(\frac{y}{x}\right)^2} = \frac{1}{xy} e^{-x^2 - 3\left(\frac{y}{x}\right)^2}.$$

Then

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{xy} e^{-x^2-3\left(\frac{y}{x}\right)^2}}{\frac{1}{x} e^{-x^2}} = \frac{1}{y} e^{-3\left(\frac{y}{x}\right)^2}.$$

$$f_Y(y) = \int_0^\infty f_{XY}(x,y)dx = \frac{1}{y} \int_0^\infty \frac{1}{x} e^{-x^2-3\left(\frac{y}{x}\right)^2} dx = \frac{1}{y} K_0\left(2\sqrt{3}|y|\right),$$

where $K_0\left(2\sqrt{3}|y|\right)$ is the modified Bessel function of the second kind.