Answer on Question #48640 - Math – Calculus

Find second order derivatives of:

1. $y = x^3 + \tan x$
2. $y = \log_a x$
3. $x = \arctan y$

Solution.

1. Use derivative of sum of functions: $y' = (x^3 + \tan x)' = 3x^2 + \frac{1}{\cos^2 x}$;

   To differentiate term $\frac{1}{\cos^2 x}$, we apply the chain rule.

   The second-order derivative is

   $$y'' = \left(3x^2 + \frac{1}{\cos^2 x}\right)' = 6x - \frac{2}{\cos^3 x}\cdot(-\sin x) = 6x + \frac{2\cos x \sin x}{\cos^4 x} = 6x + \frac{\sin 2x}{\cos^4 x}.$$

   Answer: $y'' = 6x + \frac{\sin 2x}{\cos^4 x}$.

2. $y' = (\log_a x)' = (\frac{\ln x}{\ln a})' = \frac{1}{x\ln a}$;

   The second-order derivative is

   $$y'' = \left(\frac{1}{x\ln a}\right)' = \frac{-\ln a}{x^2\ln^2 a} = -\frac{1}{x^2\ln a}.$$ If $a = e$, then $y'' = -\frac{1}{x^2}$

   Answer: $y'' = -\frac{1}{x^2\ln a}$.

3. The function $x = \arctan y$ is the inverse to the function $y = \tan x$. Therefore,

$$x'_y = \frac{1}{y'_x} = \frac{1}{(\tan x)'} = \cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2} .$$

$$x''_{yy} = -\frac{y''_{xx}}{(y'_x)^2}\cdot\frac{1}{y'_x} = -\frac{y''_{xx}}{(y'_x)^3} = \frac{-\frac{\sin 2x}{\cos^4 x}}{\left(\frac{1}{\cos^2 x}\right)^3} = -\sin 2x\cos^2 x = -\frac{2\tan x}{1+\tan^2 x}\cos^2 x = -\frac{2\tan x}{(1+\tan^2 x)^2} =$$

$$= -\frac{2y}{(1+y^2)^2}$$

Other method: let y be independent variable, x be dependent variable. Then

$x'_y = (\arctan(y))' = \frac{1}{1+y^2}$. Using the chain rule, obtain the second-order derivative:

$$x''_{yy} = \left(\frac{1}{1+y^2}\right)' = -\frac{1}{(1+y^2)^2}\cdot 2y = -\frac{2y}{(1+y^2)^2}$$

Answer: $(\text{arctg}\, y)'' = -\frac{2y}{(1+y^2)^2}$.