Answer on Question #48162 – Math – Trigonometry

Find all the values of θ between 0 and 2π for which cos θ/2 = √(1+sin⁡θ) - √(1- sin⁡θ) is true.

Solution.

Method 1:

Denote $x = \frac{\theta}{2}$. Hence:

$$\theta \in [0,2\pi] \Longrightarrow \frac{\theta}{2} \in [0,\pi], \qquad x \in [0,\pi] \Longrightarrow \sin x \geq 0;$$

$$1 + \sin\theta = \sin^2 x + 2\sin x\cos x + \cos^2 x = (\sin x + \cos x)^2;$$

$$1 - \sin\theta = \sin^2 x - 2\sin x\cos x + \cos^2 x = (\sin x - \cos x)^2;$$

$$\cos\frac{\theta}{2} = \sqrt{1+\sin\theta} - \sqrt{1-\sin\theta} \Longrightarrow \cos x = |\sin x + \cos x| - |\sin x - \cos x| \Longrightarrow$$

$$\Longrightarrow \cos x = \sqrt{2}\left(\left|\sin\left(x+\frac{\pi}{4}\right)\right| - \left|\sin\left(x-\frac{\pi}{4}\right)\right|\right);$$

$$\sin\left(x+\frac{\pi}{4}\right) \geq 0 \text{ on } \left[0,\frac{3\pi}{4}\right];$$

$$\sin\left(x+\frac{\pi}{4}\right) \leq 0 \text{ on } \left[\frac{3\pi}{4},\pi\right];$$

$$\sin\left(x-\frac{\pi}{4}\right) \geq 0 \text{ on } \left[\frac{\pi}{4},\pi\right];$$

$$\sin\left(x-\frac{\pi}{4}\right) \leq 0 \text{ on } \left[0,\frac{\pi}{4}\right];$$

Consider 3 cases:

1) $x \in [0,\frac{\pi}{4}]$:

$$\cos x = \sqrt{2}\left(\sin\left(x+\frac{\pi}{4}\right) + \sin\left(x-\frac{\pi}{4}\right)\right) \Longrightarrow \cos x = 2\sin x \Longrightarrow$$

$$\Longrightarrow \sqrt{5}\sin\left(x - \arcsin\frac{1}{\sqrt{5}}\right) = 0 \Longrightarrow x = \arcsin\frac{1}{\sqrt{5}} \Longrightarrow \theta = 2\arcsin\frac{1}{\sqrt{5}};$$

2) $x \in \left[\frac{\pi}{4},\frac{3\pi}{4}\right]$:

$$\cos x = \sqrt{2}\left(\sin\left(x+\frac{\pi}{4}\right) - \sin\left(x-\frac{\pi}{4}\right)\right) \Longrightarrow \cos x = 2\cos x \Longrightarrow \cos x = 0 \Longrightarrow x = \frac{\pi}{2} \Longrightarrow$$

$$\Longrightarrow \theta = \pi;$$

3) $x \in \left[\frac{3\pi}{4},\pi\right]$:

$$\cos x = \sqrt{2}\left(-\sin\left(x+\frac{\pi}{4}\right) - \sin\left(x-\frac{\pi}{4}\right)\right) \Longrightarrow \cos x = -2\sin x \Longrightarrow$$

$$\Rightarrow \sqrt{5}\sin\left(x+\arcsin\frac{1}{\sqrt{5}}\right)=0 \Rightarrow x=-\arcsin\frac{1}{\sqrt{5}}+\pi \Rightarrow \theta = 2\pi - 2\arcsin\frac{1}{\sqrt{5}}.$$

Answer.

$\left\{\pi, 2\arcsin\frac{1}{\sqrt{5}}, 2\pi - 2\arcsin\frac{1}{\sqrt{5}}\right\}.$

Method 2.

cos θ/2 = √(1+sin θ) - √(1- sin θ)

If $0 \le \theta \le 2\pi$, then $0 \le \frac{\theta}{2} \le \pi$.

Let $0 \le \theta \le \pi$, it implies $0 \le \frac{\theta}{2} \le \frac{\pi}{2}$, $cos\left(\frac{\theta}{2}\right) \ge 0$. Consider

$\sqrt{\frac{1+\cos(\theta)}{2}} = \sqrt{1+\sin(\theta)} - \sqrt{1-\sin(\theta)},$

Raise both sides to the second power:

$\frac{1+\cos(\theta)}{2} = 1+\sin(\theta)+1-\sin(\theta) - 2|cos\theta|,$

If $\cos(\theta) \ge 0$, $\theta \in \left[0;\frac{\pi}{2}\right]$, then $\frac{1+\cos(\theta)}{2} = 2 - 2\cos(\theta),$

$1+\cos(\theta) = 4 - 4cos(\theta)$, $5cos(\theta) = 3$, $cos(\theta) = \frac{3}{5}$,

$sin(\theta) = \sqrt{1-cos^2(\theta)} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$.

If $\theta \in \left[\frac{\pi}{2}; \pi\right]$ then $\cos(\theta) \le 0$, $\frac{1+\cos(\theta)}{2} = 2+2\cos(\theta)$, $1+\cos(\theta) = 4+4\cos(\theta)$,

$\cos(\theta) = -1$, $\sin(\theta) = 0$, $\theta = \pi$.

Let $\pi \le \theta \le 2\pi$, it implies $\frac{\pi}{2} \le \frac{\theta}{2} \le \pi$, $cos\left(\frac{\theta}{2}\right) \le 0$. Consider

$-\sqrt{\frac{1+\cos(\theta)}{2}} = \sqrt{1+\sin(\theta)} - \sqrt{1-\sin(\theta)},$

Raise both sides to the second power:

$\frac{1+\cos(\theta)}{2} = 1+\sin(\theta)+1-\sin(\theta) - 2|cos\theta|,$

If $\cos(\theta) \ge 0$, $\theta \in \left[\frac{3\pi}{2}; 2\pi\right]$, then $\frac{1+\cos(\theta)}{2} = 2 - 2\cos(\theta),$

$1+\cos(\theta) = 4 - 4cos(\theta)$, $5cos(\theta) = 3$, $cos(\theta) = \frac{3}{5}$,

$sin(\theta) = -\sqrt{1-cos^2(\theta)} = -\sqrt{1-\frac{9}{25}} = -\frac{4}{5}$.

If $\cos(\theta) \le 0, \theta \in \left[\pi; \frac{3\pi}{2}\right]$, then $\frac{1+\cos(\theta)}{2} = 2 + 2\cos(\theta)$, $1 + \cos(\theta) = 4 + 4\cos(\theta)$,
$\cos(\theta) = -1, \sin(\theta) = 0, \theta = \pi$.

Answer.

$\left\{\pi, 2\arcsin\frac{1}{\sqrt{5}}, 2\pi - 2\arcsin\frac{1}{\sqrt{5}}\right\}$.