Answer on Question #48119 – Math - Abstract Algebra

Problem.

Factorise 10 in two ways in Z[√−6]. Hence, show that Z[√−6] is not a UFD

Solution.

$$10 = (2 + \sqrt{-6})(2 - \sqrt{-6})$$

and

$$10 = 2 \cdot 5$$

Now we will prove that $2 - \sqrt{-6}$, $2 + \sqrt{-6}$, 2 and 5 are irreducible elements of $\mathbb{Z}\left[-\sqrt{6}\right]$.

- Suppose that there exist $a, b \neq 1{,}2$ such that $ab = 2$. Then $2 = N(2) = N(a)N(b)$. Hence $N(a) = N(b) = 2$.
  If $a = m + n\sqrt{-6}$. Therefore from $N(a) = 2$. $m^2 + 6n^2 = 2$. Then $n = 0$ and $m^2 = 2$. We obtain contradiction. Hence 2 is irreducible.
- Suppose that there exist $a, b \neq 1{,}5$ such that $ab = 5$. Then $25 = N(2) = N(a)N(b)$. Hence $N(a) = N(b) = 5$.
  If $a = m + n\sqrt{-6}$. Therefore from $N(a) = 5$. $m^2 + 6n^2 = 5$. Then $n = 0$ and $m^2 = 5$. We obtain contradiction. Hence 5 is irreducible.
- Suppose that there exist $a, b \neq 1, 2 + \sqrt{-6}$ such that $ab = 2 + \sqrt{-6}$. Then $10 = N\left(2 + \sqrt{-6}\right) = N(a)N(b)$. Hence $N(a) = 2$ and $N(b) = 5$ (w. l. o .g.). In two previous parts we prove that for all $c \in \mathbb{Z}[-6]$: $N(c) \neq 2$ and $N(c) \neq 5$. We obtain contradiction. Hence $2 + \sqrt{-6}$ is irreducible.
- Suppose that there exist $a, b \neq 1, 2 - \sqrt{-6}$ such that $ab = 2 - \sqrt{-6}$. Then $10 = N\left(2 - \sqrt{-6}\right) = N(a)N(b)$. Hence $N(a) = 2$ and $N(b) = 5$ (w. l. o .g.). In two previous parts we prove that for all $c \in \mathbb{Z}[-6]$: $N(c) \neq 2$ and $N(c) \neq 5$. We obtain contradiction. Hence $2 - \sqrt{-6}$ is irreducible.

Therefore $2 - \sqrt{-6}$, $2 + \sqrt{-6}$, 2 and 5 are irreducible elements of $\mathbb{Z}\left[-\sqrt{6}\right]$.

Suppose that we have two distinct factorizations of 10 and $\mathbb{Z}\left[-\sqrt{6}\right]$ is UFD. Then all factors are pairwise associated. Hence $N(2) = N(2 + \sqrt{-6})$ or $N(2) = N(2 - \sqrt{-6})$. Therefore $4 = 10$. We obtain contradiction. Hence $\mathbb{Z}\left[-\sqrt{6}\right]$ is not UFD.