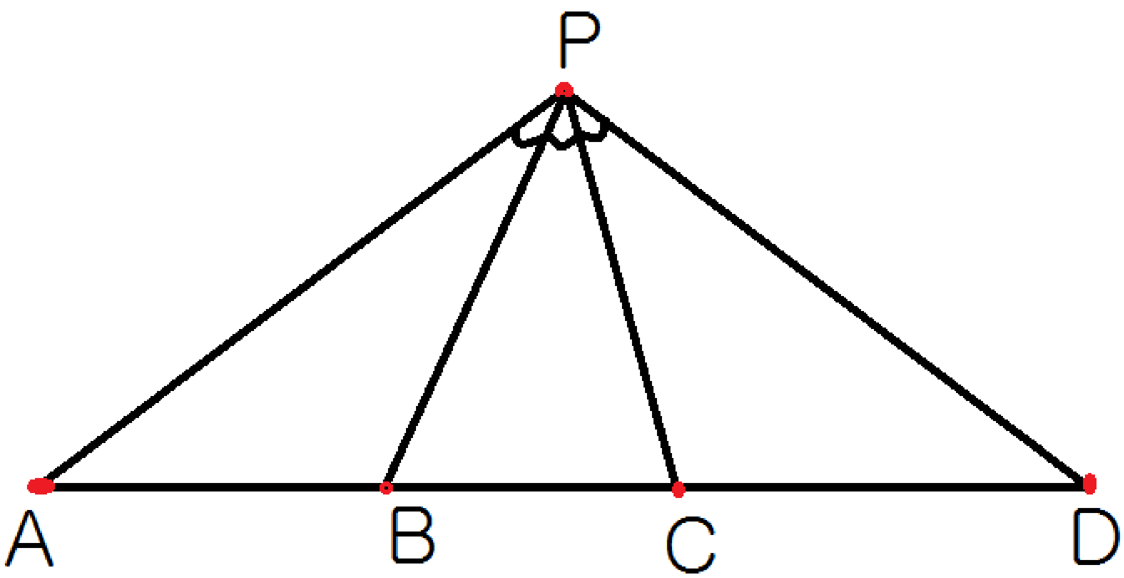
Answer on Question #48117 – Math – Geometry

A, B, C, D are four points taken on a line in that order. AB = a, BC = b, CD = c. P is a point outside this line. Measures of angles APB = BPC = CPD = θ. Prove that cos^2(θ) = (a + b)(b + c)/4ac.

Solution.



Denote $AP = x, BP = y, CP = z, DP = t$.

By the bisector theorem we have:

$$\frac{AB}{AP} = \frac{BC}{CP} \Longrightarrow \frac{a}{x} = \frac{b}{z} \Longrightarrow z = \frac{b}{a}x;$$

$$\frac{BC}{BP} = \frac{CD}{PD} \Longrightarrow \frac{b}{y} = \frac{c}{t} \Longrightarrow t = \frac{c}{b}y;$$

$$S_{APC} = S_{APB} + S_{BPC} \Longrightarrow xz\frac{\sin 2\theta}{2} = xy\frac{\sin\theta}{2} + yz\frac{\sin\theta}{2} \Longrightarrow$$

$$\Longrightarrow xy + yz = 2xz\cos\theta \Longrightarrow y = \frac{2xz\cos\theta}{x+z} = \frac{2x\frac{b}{a}x\cos\theta}{x+\frac{b}{a}x} = \frac{2bx\cos\theta}{a+b};$$

Hence:

$$t = \frac{c}{b}y = \frac{2cx\cos\theta}{a+b};$$

$$S_{APD} = S_{APB} + S_{BPC} + S_{CPD} \Longrightarrow \frac{xt\sin 3\theta}{2} = \frac{xy\sin\theta}{2} + \frac{yz\sin\theta}{2} + \frac{zt\sin\theta}{2} \Longrightarrow$$

$$\Longrightarrow xt(3\sin\theta - 4\sin^3\theta) = \sin\theta\,(xy + yz + zt) \Longrightarrow$$

$$\Longrightarrow xt(3-4\sin^2\theta) = xy+yz+zt \Longrightarrow 4\cos^2\theta - 1 = \frac{xy+yz+zt}{xt} \Longrightarrow$$

$$\Longrightarrow \cos^2\theta = \frac{xy+yz+zt+xt}{4xt} = \frac{(x+z)(y+t)}{4xt} = \frac{\left(x+\frac{b}{a}x\right)\left(y+\frac{c}{b}y\right)}{4x\frac{2cx\cos\theta}{a+b}} =$$

$$= \frac{\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\frac{2bx\cos\theta}{a+b}}{\frac{8cx\cos\theta}{a+b}} = \frac{\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)b}{4c} = \frac{(a+b)(b+c)}{4ac}.$$